Algebra Preliminary Problems

Fall 2022

1. Let $f\colon G \to H$ be a homomorphism between groups G and H and let X be a subset of G. Prove that $f(\langle X\rangle) = \langle f(X)\rangle$. Here $\langle X\rangle$ denotes the subgroup generated by X.

2. Let p be a prime and n a positive integer with $n < p^2$. Find (with proof) a Sylow p-subgroup of the symmetric group $\mathrm{Sym}(n)$.

3. Let p be a prime and F a field of characteristic p. Prove that the rings $F[X]/(X^p-1)$ and $F[X]/(X^p)$ are isomorphic.

4. Let A be a 3×3 real matrix such that $A^tA = I_3$ and $\det(A) = 1$. Prove that there is a nonzero vector $v \in \mathbb{R}^3$ such that $Av = v$.

5. Let $V = \mathbb{R}^4$ and consider the symmetric bilinear form on V :

$$\langle \vec{x}, \vec{y}\rangle = x_0y_0 - x_1y_1 - x_2y_2 - x_3y_3,$$

for $\vec{x} = (x_0, x_1, x_2, x_3)$ and $\vec{y} = (y_0, y_1, y_2, y_3)$. For any subspace W of V, we denote $W^\perp = \{\vec{v} \in V : \langle \vec{v}, \vec{w}\rangle = 0 \ \text{ for all } \vec{w} \in W\}$.

   (a) Find a nonzero subspace W such that $W \subseteq W^\perp$.

   (b) Prove or disprove : For any subspace W of V, we have $(W^\perp)^\perp = W$.

6. Let p be a prime number, $F = \mathbb{F}_p$ the field with p elements, and $V = F^2$. Let $\mathcal{L}$ denote the set of 1-dimensional F-subspaces W of V.

   (a) Prove that the natural action of the group $GL_2(F)$ on $\mathcal{L}$:

$$g \cdot W = \{gw : w \in W\}$$

   is transitive.

   (b) Prove or disprove : (when $p = 3$) the quotient group of $GL_2(\mathbb{F}_3)$ by $\{\pm I_2\}$ is isomorphic to the symmetric group on 4 letters.

7. Let $\alpha_1, ..., \alpha_k$ be elements of an extension field K of F, and assume that they are all algebraic over F. Prove that $F(\alpha_1, ..., \alpha_k) = F[\alpha_1, ..., \alpha_k]$.

8. Let p be a prime, and let a be a rational number which is not a pth power of any rational number. Let K be a splitting field of $x^p - a$ over $\mathbb{Q}$.

   (a) Prove that $x^p - a$ is irreducible in $\mathbb{Q}[x]$.

   (b) Show that the Galois group $Gal(K/\mathbb{Q})$ is isomorphic to the group of transformations of $\mathbb{Z}/p\mathbb{Z}$ of the form $y \to ky + l$ where $k, l \in \mathbb{Z}/p\mathbb{Z}$ and $k \neq 0$.

9. Prove or find a counterexample: For every $n \in \mathbb{N}$, there is a finite extension of $\mathbb{Q}$ whose Galois group is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{\oplus n}$.