ALGEBRA PRELIMINARY EXAM — FALL 2019

Problem 1. Let G be a group and let H be a subgroup of finite index. Show that there exists a normal subgroup $N \trianglelefteq G$ of finite index which is contained in H.

Problem 2. Let G be a finite group. Prove that the following are equivalent:

(1) Every Sylow subgroup of G is normal.
(2) G is isomorphic to the direct product of its Sylow subgroups.

Problem 3. Let $\mathbb{F}_3 = \mathbb{Z}/3\mathbb{Z}$ be the field with three elements and let I be the ideal of $\mathbb{F}_3[X]$ generated by the element $X^3 + X^2 + 2X + 1$. Show that $\mathbb{F}_3[X]/I$ is a field.

Problem 4. Let K be a splitting field of $x^3 - 2$ over $\mathbb{Q}$. Determine the subfields of K which are Galois over $\mathbb{Q}$.

Problem 5. Let K/F be a finite Galois extension with Galois group $G = \mathrm{Aut}(K/F)$. Suppose that $E \subseteq K$ is the fixed field of the subgroup $H \leq G$. Show that for any $\sigma \in G$, the fixed field of $\sigma H \sigma^{-1}$ is $\sigma(E)$. [*Hint:* Show first that $\sigma(E)$ is contained in the fixed field of $\sigma H \sigma^{-1}$ and then use the Galois correspondence to argue that we have equality.]

Problem 6. Let F be a field of characteristic $p > 0$. The following statements are equivalent:

(1) Every irreducible polynomial $p(x) \in F[x]$ is separable.
(2) The Frobenius endomorphism $\varphi\colon F \to F$, defined by $\alpha \mapsto \alpha^p$, is surjective.

Give a proof for either (1) $\implies$ (2) or (2) $\implies$ (1).

Problem 7. Let $n \geq 1$ be any integer and A any $n \times n$ matrix with entries in the integers $\mathbb{Z}$. Prove or disprove: There necessarily exist $n \times n$ matrices K, K' and D with entries in $\mathbb{Z}$ such that

(i) $A = KDK'$;
(ii) $\det K = 1 = \det K'$; and
(iii) D is a diagonal matrix.

Problem 8. Let p be a prime number, $F = \mathbb{Z}/p\mathbb{Z}$ the field with p elements, and $V = F^4$ the 4-dimensional F-vectorspace.

Count the number of 2-dimensional F-subspaces in V.

Problem 9. Let $S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ be the unit 2-sphere and $A = C(S; \mathbb{R})$ be the ring of real-valued continuous functions on S. Consider the A-submodule of the free module A^3:

$$T = \{(a, b, c) \in A^3 : ax + by + cz = 0\},$$

in which the coordinate functions x, y, z are considered as elements of A.

Prove or disprove: T is a projective A-module.