ALGEBRA PRELIMINARY EXAM — WINTER 2020

Problem 1. Show that there is no simple group of order 8128.

Problem 2. Let G be a group of order n. Show that there are two subgroups H_1 and H_2 of the symmetric group S_n, both isomorphic to G, such that $h_1 h_2 = h_2 h_1$ for all $h_1 \in H_1$ and $h_2 \in H_2$.

Problem 3. Prove the following:

(1) A finite abelian group G is either cyclic or has exponent less than $|G|$, i.e. there is some $m < |G|$ such that $g^m = e$ for all $g \in G$.
(2) The group of units of the ring $\mathbb{Z}/p\mathbb{Z}$ is cyclic.

Problem 4. Consider the real matrix

$$A = \begin{pmatrix} 2 & 1 & 3 & 5 \\ 4 & 3 & 7 & 0 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 3 & 2 \end{pmatrix},$$

regarded also as a linear endomorphism of $V = \mathbb{R}^4$. Compute the trace of $\wedge^2 A : \wedge^2 V \to \wedge^2 V$.

Problem 5. Let $M = \mathbb{Z}^3$ be the free $\mathbb{Z}$-module of rank 3. Let N be the $\mathbb{Z}$-submodule of M generated by the following 3 elements:

$$x = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}, \; y = \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}, \; \text{and} \; z = \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix},$$

and let $Q := M/N$.

(a) Compute the rank of Q.
(b) Compute the invariant factors of Q.

Problem 6. Let p be a prime number and $\mathbb{F}_p$ the field with p elements. Count the number of 2×2 matrices A with entries in $\mathbb{F}_p$ such that $A^2 = A$.

Problem 7. Show that an algebraically closed field is infinite.

Problem 8. Let K/F and L/K be field extensions. Prove or give a counterexample to each of the following statements:

(1) If K/F and L/K are normal, then so is L/F.
(2) If L/F is normal, then so is L/K.

Problem 9. Let F be a finite field, $f(x) \in F[x]$ an irreducible polynomial and α a root of $f(x)$. Show that $F(\alpha)/F$ is a Galois extension with cyclic Galois group.

Find the Galois group of (the splitting field of) $x^4 + 2x + 2$ over $\mathbb{F}_5$.