ALGEBRA PRELIM PROBLEMS, SPRING 2021

(1) Let G be a finite group, let p be a prime, let P be a Sylow p-subgroup of G, and let N be a normal subgroup of G.

(a) Show that $N \cap P$ is a Sylow p-subgroup of N and that PN/N is a Sylow p-subgroup of G/N.

(b) Give an example of a finite group G, a prime p, a Sylow p-subgroup P of G, and a subgroup H of G such that $H \cap P$ is not a Sylow p-subgroup of H. Justify your answer.

(2) Show that every group of order $n = 2^k \cdot 3^l$ with $0 \leq k \leq 3$ and $l \geq 0$ is solvable. (Please do not use Burnside's Theorem.)

(3) Denote by $\mathbb{F}_2$ the field with 2 elements.

(a) Is $\mathbb{F}_2[X]/(X^5 - 1)$ a direct product of fields? Justify your answer.

(b) Is $\mathbb{F}_2[X]/(X^6 - 1)$ a direct product of fields? Justify your answer.

(4) Let $\mathbb{F}_q$ be a finite field with q elements.

(a) Find and prove a formula for the number of 1-dimensional $\mathbb{F}_q$-subspaces of $\mathbb{F}_q^2$.

(b) Generalize this to find and prove a formula for the number of n-dimensional $\mathbb{F}_q$-subspaces of $\mathbb{F}_q^{2n}$, for all positive integers n.

(5) Let F be any field. Let n be a positive *even* integer, and let C be the $n \times n$ "checkerboard" matrix, whose entries C_{ij} are given by

$$C_{ij} = \begin{cases} 1 & \text{if } i + j \text{ is even;} \\ 0 & \text{if } i + j \text{ is odd.} \end{cases}$$

Here the indices satisfy $1 \leq i, j \leq n$. Let ϕ be the linear map from F^n to F^n represented by this matrix. (It does not matter whether one uses row or column vectors, in this problem.)

(a) Compute $\dim(\mathrm{Ker}(\phi))$ and $\dim(\mathrm{Im}(\phi))$. Justify your answer.

(b) What are the eigenvalues of ϕ? Justify your answer.

(c) What are the minimal and characteristic polynomials of ϕ? Justify your answer.

Date: May 12, 2021.

(6) Let F be a field, and let $\phi\colon V \to W$ be an injective map of F-vector spaces.

(a) Prove that $[\phi \oplus \phi]$ is an injective map from $V \oplus V$ to $W \oplus W$. Here, the linear map $[\phi \oplus \phi]$ is defined by

$$[\phi \oplus \phi]((v_1, v_2)) = (\phi(v_1), \phi(v_2)) \text{ for all } v_1, v_2 \in V.$$

(b) Prove that $[\phi \otimes \phi]$ is an injective map from $V \otimes V$ to $W \otimes W$. Here the linear map $[\phi \otimes \phi]$ is uniquely determined by the property

$$[\phi \otimes \phi](v_1 \otimes v_2) = \phi(v_1) \otimes \phi(v_2) \text{ for all } v_1, v_2 \in V.$$

(7) Let $E = \mathbb{Q}(\alpha)$, $F = \mathbb{Q}(\beta)$ be Galois extensions of $\mathbb{Q}$, with $E, F \subset \mathbb{C}$ and

$$[E : \mathbb{Q}] = [F : \mathbb{Q}] = 3 \text{ and } E \neq F.$$

Prove that $K = \mathbb{Q}(\alpha, \beta)$ is a Galois extension of $\mathbb{Q}$ and $\mathrm{Gal}(K/\mathbb{Q}) \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$.

(8) Prove or disprove the following

(a) If $K \supset F$ and $F \supset E$ are Galois extensions then $K \supset E$ is a Galois extension.

(b) If E, F are subfields of K such that both $[K : E]$ and $[K : F]$ are finite then $[K : E \cap F]$ is finite.

(9) Let F be a splitting field of $x^8 + 2 \in \mathbb{Q}[x]$. What is $\mathrm{Gal}(F/\mathbb{Q})$? Justify your answer.

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