ALGEBRA PRELIMINARY EXAM — SPRING 2023

Problem 1. Prove or disprove: For any group G and any subgroup $H \leq G$ with the index $|G : H| = n < \infty$, there exists a normal subgroup $N \trianglelefteq G$ such that $N \leq H$ and $|G : N| \leq n!$.

Problem 2. Let $G = \mathrm{Sym}(5)$ be the symmetric group on 5 letters and $H = \mathrm{Alt}(5)$ the alternating subgroup.

(a) Count with justification the number of 5-cycles in G.

(b) Prove or disprove: The H-conjugacy class of any 5-cycle $\sigma \in H$ contains exactly 12 elements.

(c) Prove or disprove: There is a 5-cycle $\sigma \in H$ such that the centraliser of $\langle \sigma \rangle$ in H is equal to the normaliser of $\langle \sigma \rangle$ in H. (Here $\langle \sigma \rangle$ denotes the subgroup generated by σ.)

Problem 3. Consider the subring $R = \{a + bi : a, b \in \mathbb{Z}\}$ of the complex numbers, where $i^2 = -1$.

(a) Prove that every ideal of R is a principal ideal.

(b) Prove or disprove: For every prime number p that is congruent to 1 modulo 4, there exist integers a and b such that $p = a^2 + b^2$.

Problem 4. Let V and W be finite-dimensional real inner product spaces.

(a) Let $T : V \to W$ be a linear map and let $T^* : W \to V$ denote the adjoint map. Prove that $\ker(T^*) = \mathrm{im}(T)^\perp$.

(b) A projection operator is a linear operator $P : V \to V$ satisfying $P^2 = P$. Show that there is a direct sum decomposition:

$$V = \ker(P) \oplus \mathrm{im}(P).$$

(Here $\ker(P)$ and $\mathrm{im}(P)$ denote the kernel and image of P, respectively.)

(c) Show that a projection operator $P : V \to V$ is self-adjoint if and only if the subspaces $\ker(P)$ and $\mathrm{im}(P)$ are orthogonal complements.

Problem 5. Consider the free abelian group $\mathbb{Z}^4$. Let $N \subseteq \mathbb{Z}^4$ be the subgroup generated by the five elements

$$\begin{pmatrix} 12 \\ -12 \\ 24 \\ 0 \end{pmatrix}, \begin{pmatrix} -12 \\ 12 \\ -12 \\ 12 \end{pmatrix}, \begin{pmatrix} 12 \\ 24 \\ 24 \\ 36 \end{pmatrix}, \begin{pmatrix} 24 \\ -24 \\ 48 \\ 0 \end{pmatrix}, \begin{pmatrix} 24 \\ 12 \\ 60 \\ 48 \end{pmatrix} \in \mathbb{Z}^4.$$

Describe the quotient abelian group $\mathbb{Z}^4/N$. More precisely, give a decomposition of $\mathbb{Z}^4/N$ as a direct sum of cyclic groups. Explicitly state the rank, invariant factors and elementary divisors. Be sure to explain your work.

Problem 6. Let A be an $n \times n$ complex matrix.

(a) Prove or disprove: A is diagonalizable if and only if its minimal polynomial has simple roots.
(b) Prove or disprove: Every nonzero nilpotent matrix is diagonalizable.
(c) Prove that the trace $\mathrm{tr}(A)$ is equal to the sum of the eigenvalues of A counted with multiplicity.

Problem 7.

(a) Find the minimal polynomial of $\sqrt{2} + \sqrt{5}$ over $\mathbb{Q}$.
(b) Does the field $\mathbb{Q}(\sqrt{2} + \sqrt{5})$ contain any solution to $x^3 - 5$? Prove that your answer is correct.

Problem 8. Let $K := \mathbb{F}_p(t)$ denote the field of rational functions in the indeterminate t with coefficients in the finite field $\mathbb{F}_p = \mathbb{Z}/p$ with p elements. Let $E := K(\alpha)$ for α a root of the polynomial $f(x) = x^p - x + t$ (in some field extension of K).

(a) Compute the degree $[E : K]$.
(b) Prove that E/K is a Galois extension.
(c) Compute the Galois group $\mathrm{Gal}(E/K)$.

(Hint: It may be helpful to notice that $x^p - x = (x + 1)^p - (x + 1)$.)

Problem 9. Let E be the field of complex rational functions $\mathbb{C}(x_1, x_2, x_3, x_4, x_5)$. Let σ denote the permutation $(x_1, x_2, x_3, x_4, x_5) \mapsto (x_2, x_3, x_4, x_5, x_1)$ and τ the permutation $(x_1, x_2, x_3, x_4, x_5) \mapsto (x_1, x_5, x_4, x_3, x_2)$. Let K denote the subfield of rational functions invariant (or symmetric) under τ and σ. How many subextensions $K \subseteq L \subseteq E$ are there? Prove that your answer is correct.