Problem 1. Let p be a prime number and let $G = GL_2(\mathbb{F}_p)$ be the multiplicative group of invertible 2×2 matrices with entries in the finite field $\mathbb{F}_p$.

(a) Compute with justification the order $|G|$.
(b) Count with justification the number of p-Sylow subgroups of G.

Problem 2. Let $G = A_4 = \mathrm{Alt}(4)$ be the alternating group on 4 letters.

(a) Prove or disprove : There is exactly 1 subgroup H of G such that $|H| = 4$.
(b) Exhibit with proof a subgroup $K \leq G$ such that $K \neq N_G(K)$ and $N_G(K) \neq G$. Here N_G denotes the normaliser in G.

Problem 3. Consider the subset of the ring $\mathbb{C}$ of complex numbers

$$R = \left\{a + b\sqrt{5}i : a, b \in \mathbb{Z}\right\}$$

(a) Prove that R is a subring of $\mathbb{C}$.
(b) Prove or disprove : every ideal of R is a principal ideal.

Problem 4. Let $A \colon \mathbb{Z}^3 \to \mathbb{Z}^3$ be the $\mathbb{Z}$-module homomorphism given by left multiplication by the matrix

$$A = \begin{pmatrix} 15 & -27 & 0 \\ -9 & 45 & 15 \\ -9 & 33 & 9 \end{pmatrix}.$$

Construct an isomorphism between the quotient group $\mathbb{Z}^3/\,\mathrm{Image}(A)$ and a direct sum of cyclic groups.

Problem 5. Let $V = \mathbb{C}^n$ be the standard n-dimensional complex vector space.

(a) Suppose that $u, v \in V$ are nonzero vectors and $c \in \mathbb{C}^\times$ is a nonzero scalar. Set $x = cu$ and $y = c^{-1}v$. Prove that $u \otimes v = x \otimes y$ in $V \otimes_{\mathbb{C}} V$.
(b) Prove the converse of the previous part. That is, if $u, v, x, y \in V$ are nonzero vectors and $u \otimes v = x \otimes y$ in $V \otimes_{\mathbb{C}} V$, then prove that there exists a nonzero scalar c such that $x = cu$ and $y = c^{-1}v$.

Problem 6. This problem concerns the matrix

$$A = \begin{pmatrix} -58 & -100 \\ 20 & 47 \end{pmatrix}.$$

Let M be the complex vector space of "column vectors" $\begin{pmatrix} x \\ y \end{pmatrix}$ with $x, y \in \mathbb{C}$.

Given a polynomial $f(t) \in \mathbb{C}[t]$, let $f(A) \in \mathrm{Mat}_{2,2}(\mathbb{C})$ be the matrix obtained by evaluating f at $t = A$ and set

$$f(t) \cdot \begin{pmatrix} x \\ y \end{pmatrix} = f(A) \begin{pmatrix} x \\ y \end{pmatrix}.$$

The right-hand side is normal matrix multiplication. This makes M into a $\mathbb{C}[t]$-module. (You do not need to prove this.)

(a) Prove or disprove : there is a nonzero element $v \in M$ such that $t \cdot v = 0$.
(b) Let N be the $\mathbb{C}[t]$-module constructed in the same way as M except replace the matrix A by its transpose A^{T}. Prove or disprove : M and N isomorphic as $\mathbb{C}[t]$-modules.

Problem 7. Let p be a prime number and let G be the Galois group of $X^p - 2$ over $\mathbb{Q}$. Prove that G is isomorphic to a semidirect product $C_p \rtimes C_{p-1}$.

Problem 8. A field extension K/F is said to be a *simple extension* if $K = F(u)$ for some $u \in K$.

(a) Let F be a finite field. Prove that every finite extension K/F is a simple extension.
(b) Let F be an infinite field. Prove that every finite extension K/F which has only finitely many intermediate fields is a simple extension.
(c) Prove that a finite extension K/F is simple if and only if it has only finitely many intermediate fields.

Problem 9.

(a) Let $f(X) = X^3 + nX + 2$ where n is an integer. Determine the (infinitely many) values of n for which f is irreducible over $\mathbb{Q}$. (Be sure to justify your answer.)
(b) Let $\overline{\mathbb{Q}}$ be the field of algebraic numbers. Prove that $\overline{\mathbb{Q}}/\mathbb{Q}$ is an infinite extension.