Algebra Preliminary Exam

Winter 2021

1. Let G be a finite group of order n. Determine the cardinality of $\mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}, G)$. Justify your answer.

2. Let G be a finite group, p a prime, P a p-subgroup of G, and N a normal subgroup of G whose order is not divisible by p. Show that $N_{G/N}(PN/N) = N_G(P)N/N$.

3. Let R be a ring and let P be an ideal of R. Consider the following two statements:

(1) For any two ideals I and J of R one has: $IJ \subseteq P \Rightarrow I \subseteq P$ or $J \subseteq P$.

(2) For any two elements a and b of R one has: $ab \in P \Rightarrow a \in P$ or $b \in P$.

(a) Show that (2) implies (1).

(b) Show that if R is commutative then (1) implies (2).

4. Let p be a prime number, $F = \mathbb{F}_p$ the field with p elements, and $K = \mathbb{F}_{p^2}$ the quadratic extension of F. Consider the Frobenius $\phi : K \to K$:

$$\phi(x) = x^p \quad \text{for all } x \in K.$$

Regard ϕ as an F-linear operator on K, and determine the Jordan canonical form of ϕ.

5. Let M and N be two finitely generated abelian groups and let $f : M \to N$ be a group homomorphism. Let $r \geq 1$ be an integer. Denote by f_r the induced group homomorphism $M/rM \to N/rN$.

(1) Prove or disprove with a counterexample: For any M, N, f and r as above, if f is injective, then f_r is necessarily injective.

(2) Prove or disprove with a counterexample: For any M, N, f as above, if f_r is surjective for every $r \geq 1$, then f is necessarily surjective.

6. Let $X = \mathbb{Z}/N\mathbb{Z}$ be a finite cyclic group. Consider the set of functions

$$V = \{ f : X \to \mathbb{C} \ \text{ functions } \}$$

equipped with the (usual) structure of complex vector space

$$(f + g)(x) = f(x) + g(x) \ \text{ and } \ (c \cdot f)(x) = c \cdot f(x)$$

for all $f, g \in V$, $c \in \mathbb{C}$ and $x \in X$. For any $f \in V$, define the linear operator C_f on V by the formula

$$C_f(g)(x) = \sum_{y \in X} f(y) g(x - y) \quad \text{for all } \ g \in V \ \text{ and } \ x \in X.$$

Prove or disprove with a counterexample : For any $f \in V$, C_f is diagonalisable.

7. Determine the Galois group of $x^5 + x - 1$ over $\mathbb{Q}$.

8. Let E be a finite extension of field F. Prove or disprove that there exists $a \in E$ such that $E = F(a)$.

9. Let p be a prime, let n be a natural number, and let F be a field that contains a primitive p^n-th root of unity. Let $a \in F$ such that $[F(a^{1/p}) : F] > 1$. Prove that $[F(a^{1/p^n}) : F] = p^n$.