Algebra Preliminary Exam, Winter 2022

1. (a) Compute the size of the conjugacy class of the n-cycle $\sigma := (1, 2, ..., n)$ in $\mathrm{Sym}(n)$ and determine the centralizer of σ in $\mathrm{Sym}(n)$. Justify your answer.

(b) Suppose that n is odd. Then $\sigma \in \mathrm{Alt}(n)$, the alternating group of degree n. Compute the size of the conjugacy class of σ in $\mathrm{Alt}(n)$. Justify your answer.

2. Let G be a finite group, acting transitively on a set X. Let $x \in X$ and let P be a Sylow p-subgroup of the stabilizer of x in G. Show that, for all $g \in N_G(P)$ and $x \in X^P$, one has $gx \in X^P$ and that the resulting action of $N_G(P)$ on X^P is again transitive. Here, X^P is the set of P-fixed points in X.

3. Compute the order of the unit group of the ring $\mathbb{F}_2[X]/(X^5 - 1)$, where $\mathbb{F}_2$ denotes the field with two elements. Justify your answer.

4. Suppose that $n \geq 2$, and $\mathbb{F}_q$ is a finite field with q elements. How many surjective linear maps are there from $\mathbb{F}_q^n$ to $\mathbb{F}_q^2$? Prove your answer.

5. Let $A = (a_{ij})$ be a square $n \times n$ matrix with entries in a field F. Suppose that there exist functions $f\colon \mathbb{N} \to F$ and $g\colon \mathbb{N} \to F$, such that

$$a_{ij} = f(i) + g(j) \text{ for all } 1 \leq i, j \leq n.$$

Prove that $n \geq 3$ implies $\det(A) = 0$.

6. Suppose that $\phi\colon \mathbb{C}^3 \to \mathbb{C}^3$ is a linear map with $\dim \mathrm{Ker}(\phi) = 1$. Let

$$[\phi \otimes \phi]\colon (\mathbb{C}^3 \otimes \mathbb{C}^3) \to (\mathbb{C}^3 \otimes \mathbb{C}^3)$$

be the unique linear map satisfying $[\phi \otimes \phi](v \otimes w) = \phi(v) \otimes \phi(w)$ for all $v, w \in \mathbb{C}^3$.

What is $\dim \mathrm{Ker}(\phi \otimes \phi)$? Prove your answer.

7. Let ζ_n be a primitive n-th root of unity. Prove that $\mathbb{Q}(\zeta_{24})/\mathbb{Q}$ is a Galois extension with Galois group isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

8. (a) Let E be an extension field of F such that $[E : F] < \infty$ and for any two subfields E_1 and E_2 containing F either $E_1 \supset E_2$ or $E_2 \subset E_1$. Prove that E is a simple extension of F.

(b) If we only remove assumption $[E : F] < \infty$ in (a), prove or disprove that E is a simple extension of F.

9. Show that $\sqrt[3]{2}$ is not contained in the cyclotomic field $\mathbb{Q}(\zeta_n)$ for any n.