Fall 2021 - Analysis Prelim Exam- Friday, October 15
University of California Santa Cruz

1. (a) Show that every open and connected subset $A \subseteq \mathbb{R}^2$ is path-connected.
   (b) Show that $X = [0,1] \times [0,1]$ in the lexicographic order topology is not path-connected.

2. Let X be a metric space, and let A and B be two closed, disjoint subsets. Show that there exists a continuous function $f : X \to [0,1]$ such that $f(x) = 0$ for $x \in A$ and $f(x) = 1$ for $x \in B$.

3. Given $E \subset \mathbb{R}$, let $\mathcal{O}_n := \{x : d(x, E) < \frac{1}{n}\}$. Let m denote the Lesbesgue measure.
   (a) Show that if E is compact, then $\lim_{n\to\infty} m(\mathcal{O}_n) = m(E)$.
   (b) Show that the conclusion is false if E is closed and unbounded, or if E is open and bounded.

4. Compute the limit
$$\lim_{n\to\infty} \int_0^1 \frac{n\ dx}{(1+nx)^2(1+x+x^2)}.$$
   Justify your calculations.

5. Let X, Y be Banach spaces and $A : X \to Y$ be a bounded linear operator. Show that the following are equivalent:
   (a) A is injective and the range of A is closed;
   (b) there exists a constant $M > 0$ such that
$$\|x\| \le M\|Ax\| \qquad \text{for all } x \in X.$$

6. Let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$, and let A_n be a sequence of bounded linear operators on H. Assume that for every $x, y \in H$, the limit $\lim_{n\to\infty} \langle y, A_n x \rangle = 0$.
   (a) Does it follow that $\lim_{n\to\infty} \|A_n\| = 0$?
   (b) Does it follow that $\sup_{n\ge 1} \|A_n\| < +\infty$?
   Provide either counterexamples or proofs.

7. Let $a > 1$. Show that the equation
$$a - z - e^{-z} = 0$$
   has exactly one solution in the right half plane $\{z \in \mathbb{C} \mid \mathrm{Re}(z) > 0\}$.

8. For $a > 0$, compute the following integral, using residue theory:
$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2 + a^2}\ dx$$