Fall 2022 - Analysis Prelim Exam - Tuesday, September 13, 2022
University of California Santa Cruz

1. Suppose that $f : X \to Y$ is a continuous and bijective map, where X is compact and Y is Hausdorff. Show that f is a homeomorphism.

2. Suppose that $f : X_0 \to Y$ is a uniformly continuous map, where X_0 is a dense subspace of a metric space X and Y is a complete metric space. Show that f admits a unique continuous extension $\overline{f} : X \to Y$ such that $\overline{f}$ is also uniformly continuous.

3. Let $(X, \mathcal{M})$ be a measurable space, and suppose $f_n \colon X \to [-\infty, \infty]$ is measurable for every $n \in \mathbb{N}$.

   (a) Show that $\overline{f}(x) := \limsup_{n\to\infty} f_n(x) = \inf_{n\geq 1} \sup_{k>n} f_k(x)$ is a measurable function.

   (b) Show that if $\{f_n\}_{n=1}^{\infty}$ converges pointwise to a function $f(x)$, then f is measurable.

4. Use convergence theorems from class to study the limit, as $n \to \infty$, of each of the following integrals:

$$\text{(a)} \quad I_n = \int_0^\infty (1+x)^{-2n} \cos x \; dx, \qquad\qquad \text{(b)} \quad J_n = n \int_0^\infty (1+x)^{-2n} \cos x \; dx.$$

5. Let X, Y be complex Banach spaces. A map $B : X \times Y \to \mathbb{C}$ is called a bilinear form, if $B(x, y)$ is linear in x for every $y \in Y$ and linear in y for every $x \in X$. It is called bounded if there exists a constant $M > 0$ such that

$$|B(x, y)| \leq M \cdot \|x\| \cdot \|y\| \qquad \text{for every } x \in X,\ y \in Y.$$

   Show that there exists a one-to-one correspondence between bounded bilinear forms $B : X \times Y \to \mathbb{C}$ and bounded linear operators $A : X \to Y^*$, where Y is the dual space of Y.

6. Let X and Y be Banach spaces, let $L(X, Y)$ stand for the set of all bounded linear operators $T : X \to Y$, which itself is a Banach space with the operator norm. Assume that the operator $T_0 \in L(X, Y)$ has an inverse, i.e., there exists $S_0 =: (T_0)^{-1} \in L(Y, X)$ such that $S_0 T_0 = I_X$ and $T_0 S_0 = I_Y$, where I_X and I_Y are the identity operators on X and Y, respectively.

   Show that there exist an $\epsilon > 0$ such that for $U = \{T \in L(X, Y) \,:\, \|T - T_0\| < \epsilon\}$, the map

$$\iota : T \in U \mapsto T^{-1} \in L(Y, X)$$

   is well-defined and continuous.

7. Suppose f and g are holomorphic in a region containing the disc $|z| \leq 1$. Suppose that f has a simple zero at $z = 0$ and vanishes nowhere else in $|z| \leq 1$. Let

$$f_\varepsilon(z) = f(z) + \varepsilon g(z).$$

   Show that if ε is sufficiently small, then $f_\varepsilon(z)$ has a unique zero in $|z| \leq 1$.

8. Suppose f is a holomorphic function on a connected open set, and $u = \mathrm{Re}(f)$. Prove that if the product $u\overline{f}$ is holomorphic, then f must be a constant function.