Spring 2021 - Analysis prelim - University of California Santa Cruz

1. Let $X = \prod_{n=1}^{\infty}[0,1] = [0,1]^{\mathbb{N}}$ be the countable direct product of the closed intervals $[0,1] \subseteq \mathbb{R}$ and consider the subset
$$S = \Big\{(x_n) \in X \ : \ \exists N,\ \forall n \geq N : x_n = 0\Big\}.$$
   (a) Show that if X is considered with the product topology, then the closure of S is X.
   (b) Show that if X is considered with the box topology, then S is closed in X.

2. Given a topological space X, and $\mathbb{Z}$ in the standard topology, define the property
$$P(X)\text{: “}\textit{Every continuous function } f : X \to \mathbb{Z} \textit{ is constant.}\text{”}$$
   (a) If $\mathbb{R}$ is equipped with the standard topology, show that $P(\mathbb{R})$ is true.
   (b) For a general topological space X, find and prove an if-and-only-if characterization of $P(X)$ is terms of a topological property of X.

3. Let $A \subset \mathbb{R}$ a measurable set with $m(A) < \infty$. (m: Lebesgue measure)
   (a) Show that the function $\varphi : \mathbb{R} \to [0,\infty)$ defined by $\varphi(x) := m(A \cap (-\infty, x])$ is continuous.
   (b) Prove that there exists $x \in \mathbb{R}$ such that $m(A \cap (-\infty, x)) = m(A \cap (x, \infty))$.

4. Let $f_1, f_2, \ldots$ and g be functions in $L^1(\mathbb{R})$ and put $E_n := \{x \in \mathbb{R} : |f_n(x)| > |g(x)|\}$. Suppose that $f_n(x) \to g(x)$ for almost every x and that $\int_{E_n} |f_n(x)|\, dx \to 0$ as $n \to \infty$. Prove that
$$\lim_{n\to\infty} \int_{\mathbb{R}} |f_n(x) - g(x)|\, dx = 0.$$

5. Let X and Y be two Banach spaces. A sequence of bounded linear operators $A_n \in L(X,Y)$ is said to converge weakly to a linear operator A (from X to Y) if for all $x \in X$ and all $\phi \in Y^*$ the sequence $\phi(A_n x)$ converges to $\phi(Ax)$. Assuming A_n converges weakly to A, show that $\sup_{n\geq 1} \|A_n\| < +\infty$ and that the operator A is bounded.

6. Let X be the set of all continuously differentiable functions $f : [-1,1] \to \mathbb{R}$, and notice that
$$\|f\|_X := \sup_{x\in[-1,1]} |f(x)|$$
makes X a normed space (in fact, $X \subseteq C([-1,1])$). Decide whether the following linear functionals on X are bounded, and if so, evaluate their norm:
$$\phi_1(f) = f(0), \qquad \phi_2(f) = \int_{-1}^{1} \operatorname{sign}(x) f(x)\, dx, \qquad \phi_3(f) = f'(0), \qquad \phi_4(f) = \sum_{n=1}^{\infty} \frac{1}{2^n} f\left(\frac{1}{n}\right).$$

7. Let $\Omega \subset \mathbb{C}$ be open and suppose that $\{f_n\}_{n=1}^{\infty}$ is a sequence of holomorphic functions on Ω that converges uniformly to $f : \Omega \to \mathbb{C}$. Show that for any $\delta > 0$ we have that $f_n' \to f'$ uniformly on the set
$$K_\delta := \{z \in \Omega : \overline{B_\delta(z)} \subset \Omega\}.$$

8. Use the residue theorem to compute
$$\int_0^{\infty} \frac{x^{1/3}}{1+x^2} dx\,.$$