Preliminary Examination Analysis 2022 May 27

1. A topological space X is said to be sequentially compact if every sequence of points in X has a subsequence that converges to a point in X. Assume X is a metric space. Show that X is sequentially compact if every open cover has a finite subcover.

2. Suppose that $\{f_n(x)\}$ is a sequence of continuously differentiable functions on $[a, b]$ such that

$$\max_{x\in[a,b]} |f_n(x)| \leq M \text{ and } \max_{x\in[a,b]} |f_n'(x)| \leq M$$

for some number $M > 0$ and all n. Prove that $\{f_n(x)\}$ has a subsequence that converges uniformly in $[a, b]$.

3. On a measureable space $(X, \mathcal{M})$, a positive measure ν is *absolutely continuous* with respect to a positive measure μ ($\nu \ll \mu$ for short) if "for all $E \in \mathcal{M}$, $\mu(E) = 0$ implies $\nu(E) = 0$".

a. Show that this definition is equivalent to "for all $\varepsilon > 0$, there exists $\delta > 0$ such that $\mu(E) < \delta$ implies $\nu(E) < \varepsilon$".

b. Show that if a family $\{\nu_n\}_{n\geq 1}$ of positive measures satisfy $\nu_n \ll \mu$ for all n, then the measure ν_∞, defined as $\nu_\infty(E) := \sum_{n\geq 1} \nu_n(E)$ for $E \in \mathcal{M}$, satisfies $\nu_\infty \ll \mu$.

4. Compute the following limit, justifying calculations.

$$\lim_{n\to\infty} \sqrt{n} \int_0^1 (1 + nx^2)(1 + x^2)^{-n} \, dx.$$

Express your result in terms of the Gamma function $\Gamma(s) := \int_0^\infty t^{s-1}e^{-t} \, dt$.

5. The operator M_a of multiplication by a given sequence $a = \{a_n\}_{n=1}^{\infty}$ is defined as follows:

$$M_a : \{x_n\}_{n=1}^{\infty} \mapsto \{a_n x_n\}_{n=1}^{\infty}.$$

Assume that $a = \{a_n\}_{n=1}^{\infty}$ is chosen such that $M_a x = y \in \ell^2$ whenever $x \in \ell^2$.

(a) Prove that the linear operator M_a is bounded on ℓ^2.
(b) Prove that $a \in \ell^{\infty}$, i.e., a is a bounded sequence.

6. Let H be a real or complex Hilbert space, and $A \in L(H)$ be a bounded linear operator on H.

(a) Show that if $\langle x, Ax \rangle = 0$ for all $x \in H$, then $A = 0$.
(b) Show that if $|\langle x, Ax \rangle| \geq \|x\|^2$ for all $x \in H$, then A is injective and has a closed range.

7. Suppose γ is a simple, closed, continuously differentiable curve in $\mathbb{C}$. What are all the possible values of $\displaystyle\int_{\gamma} \frac{z}{z^2+1} dz$ for different choices of γ? Explain.

8. Prove that if $0 < |z| < 1$, then $\dfrac{1}{4}|z| < |1 - e^z| < \dfrac{7}{4}|z|$.