Analysis Preliminary Exam, Math @ UCSC, Spring 2023

1. Suppose that $\{f_n(x)\}$ is a sequence of continuous functions that converges pointwisely to a function $f(x)$ on a compact metric space X. Assume that $\{f_n(x)\}$ is uniformly equi-continuous, i.e. for any $\epsilon > 0$, there is a $\delta > 0$ such that, for any n,

$$|f_n(x) - f_n(y)| < \epsilon$$

   $x, y \in X$ and $d(x, y) < \delta$. Show that $\{f_n(x)\}$ actually uniformly converges and the limit function $f(x)$ is also continuous.

2. Let $f : X \to Y$ be a bijective continuous map, where X is a compact topological space and Y is Hausdorff. Show that f is in fact a homeomorphism. Here is the definition of compactness for this problem: a topological space A is compact if any open cover $\{U_\alpha\}$ of A has a finite subcover.

3. Suppose that $f : [0, 1] \to \mathbb{R}$ is in L^1 and for each $n \in \mathbb{N}$ we define

$$f_n(x) := n \int_{k/n}^{(k+1)/n} f(y)dy$$

   for $x \in [k/n, (k + 1)/n)$ and $0 \le k \le n - 1$. Prove that $f_n \to f$ in L^1.

4. Let f and g be real-valued integrable functions on a measure space $(X, \mathcal{M}, \mu)$ and for any $t \in \mathbb{R}$ define

$$F_t := \{x \in X : f(x) > t\} \quad \text{and} \quad G_t := \{x \in X : g(x) > t\} .$$

   Prove that

$$\int_X |f - g| \, d\mu = \int_{-\infty}^{\infty} \mu \left(F_t \Delta G_t \right) dt ,$$

   where $F_t \Delta G_t := (F_t \setminus G_t) \cup (G_t \setminus F_t)$. Note that $(X, \mathcal{M}, \mu)$ is an arbitrary measure space.

5. Let X be a Banach space and consider the unit sphere

$$S = \{ \phi \in X^* \, : \, \|\phi\| = 1\}$$

   in the dual space X^*. Show that S is not closed in the weak*-topology of X^* by showing that $0 \in X^*$ is an accumulation point of S in the weak*-topology.

6. Let $\{A_\omega\}_{\omega \in \Omega}$ be a family of bounded linear operators $A_\omega : X \to Y$ where X and Y are Banach spaces. Suppose that for each $x \in X$, $\phi \in Y^*$,

$$\sup_{\omega \in \Omega} |\phi(A_\omega x)| < \infty.$$

   Show that

$$\sup_{\omega \in \Omega} \|A_\omega\|_{L(X,Y)} < \infty.$$

7. (a) State Liouville's theorem.

   (b) Suppose f is entire and such that there exist a constant $C > 0$ and $p \in \mathbb{N}$ such that $|f(z)| \leq C|z|^p$ for every z with $|z| \geq 1$. Prove that f is a polynomial of degree at most p.

8. Evaluate the integrals

$$\int_0^\infty e^{-ax} \cos(bx)\, dx \quad \text{and} \quad \int_0^\infty e^{-ax} \sin(bx)\, dx, \qquad a > 0,$$

by integrating e^{-Az}, $A = \sqrt{a^2 + b^2}$, over an appropriate sector with angle ω with $\cos \omega = \frac{a}{A}$.

Carefully write down any limiting process involved for full credit.