Analysis prelim - Spring 2024 - UC Santa Cruz - 6/14/2024

1. Let (X, d) be a metric space. For $\rho \geq 0$ and $x \in X$, let $B_\rho(x) := \{y \in X,\ d(x,y) < \rho\}$. Declare $U \subseteq X$ to be open if for every $x \in U$, there is $\rho > 0$ such that $B_\rho(x) \subset U$.

   (a) Show that for any $\rho > 0$ and $x \in X$, $B_\rho(x)$ is open.

   (b) Show that $\tau := \{U \subseteq X : U \text{ is open}\}$ is a topology.

2. Let $f\colon X_0 \to Y$ be a uniformly continuous function, where Y is a complete metric space and X_0 is a dense subset of a metric space X.

   (a) Show that f has a unique continuous extension to a function $\hat{f}\colon X \to Y$.

   (b) Show that $\hat{f}$ is also uniformly continuous.

3. Let (X, Λ, μ) be a measure space. Suppose that $\{A_i\}_{i=1}^\infty$ is a sequence of measurable subsets such that $\mu(A_i) \leq 2^{-i}$ for every $i \geq 1$. Show that

$$\mu\Big(\bigcap_{n=1}^\infty \bigcup_{i=n}^\infty A_i\Big) = 0.$$

4. Let (X, Λ, μ) be a measure space. Suppose that $f : X \to \mathbb{R}$ is a measurable function such that $\int_X |f| d\mu < \infty$. Show that, for each $\epsilon > 0$, there exists $\delta > 0$ such that

$$\int_A |f| d\mu < \epsilon$$

   for any measurable subset $A \subset X$ with $\mu(A) < \delta$.

5. Let $A : H \to H$ be a linear operator on a Hilbert space H satisfying $\langle Ax, y\rangle = \langle x, Ay\rangle$ for all $x, y \in H$. Show that A is a bounded operator. (Hint: first show that A has a closed graph.)

6. Let X be a norm space with norm $\|\cdot\|$, let M be a closed subspace of X. Consider the quotient space X/M, i.e., the set of all $[x] = x + M$, $x \in X$.

   (a) Prove that $\big\|[x]\big\|_* = \inf\{\, \|x - z\| \ :\ z \in M\,\}$ defines a norm on X/M, i.e., X/M is a normed space.

   (b) Prove that if X is a Banach space, then X/M is a Banach space, too.

   (c) Is the converse true ? Provide a counter-example if not true.

   Hint: you may use the following fact. A normed space $(Z, \|\cdot\|)$ is complete if and only if every absolutely convergent series converges, i.e., if $\sum_{n=1}^\infty \|z_n\| < \infty$ with $z_n \in Z$, then $\sum_{n=1}^\infty z_n$ converges in Z.

7. Show, using residue theory, the equality

$$\int_{\mathbb{R}} \frac{dx}{(x^2 + a^2)^2} = \frac{\pi}{2a^3}, \qquad a > 0.$$

   Justify any limiting process involved.

8. (a) State Liouville's theorem.

   (b) Show that if $f(z)$ is an entire function, and there is a nonempty open disk such that $f(z)$ does not attain any values in the disk, then $f(z)$ is constant.