Winter 2021 - Analysis Prelim - Friday, January 29
University of California Santa Cruz

1. Give the definitions of *compactness* and *limit point compactness* of a topological space. Show that every compact space is limit point compact. Give an example that the converse is not true.

2. Show that the image of a continuous function $f : X \to Y$ is connected if X is connected. Here X and Y are topological spaces.

3. Let X be an uncountable set and let $\mathcal{M} := \{E \subset X : \text{ either } E \text{ or } E^c \text{ is at most countable}\}$. Define $\mu\colon \mathcal{M} \to [0, \infty]$ by $\mu(E) = 0$ if E is at most countable, or $\mu(E) = 1$ if E^c is at most countable.

   (a) Prove that $\mathcal{M}$ is a σ-algebra and that μ is a measure on $\mathcal{M}$.

   (b) Prove that $\mathcal{M}$ is the σ-algebra generated by $\mathcal{E} = \{\{x\} : x \in X\}$.

   Note: For $\mathcal{E} \subset \mathcal{P}(X)$, the σ-algebra generated by $\mathcal{E}$ is the smallest σ-algebra containing $\mathcal{E}$.

4. Let $\{f_n\}$ be a sequence of Lebesgue measurable functions defined on a set $E \subset \mathbb{R}$ of finite Lebesgue measure. Show that f_n converges to zero **in measure** if and only if

$$\int_E \frac{|f_n|}{1 + |f_n|}\, dm \to 0 \qquad \text{as } n \to \infty \qquad (dm : \text{Lebesgue measure}).$$

5. Let X be a Banach space which decomposes into a direct sum of linear subspaces $X = X_1 \dot{+} X_2$, i.e., each $x \in X$ can be written as a sum $x = x_1 + x_2$ with uniquely determined $x_1 \in X_1$ and $x_2 \in X_2$. Let $P : X \to X$ be the map defined by $Px = x_1$.
   Show that P is a linear operator satisfying $P^2 = P$. Moreover, show that P is bounded if and only if both X_1 and X_2 are closed subspaces of X.

6. Show that every closed linear subspace of a reflexive Banach space is reflexive.

7. Use the residue theorem to evaluate

$$\int_0^\pi \frac{d\theta}{2 + \sin(2\theta)}\,.$$

8. Suppose that $f : \mathbb{C} \to \mathbb{C}$ is entire with $f(z) = \sum_{n=0}^\infty a_n z^n$.

   (a) Show that f has an essential singularity at infinity if $a_n \neq 0$ for infinitely many n's.

   (b) Show that if f is injective then $f(z) = a_0 + a_1 z$ with $a_1 \neq 0$.