Winter 2022 - Analysis prelim - University of California Santa Cruz

1. Suppose that $\{f_n\}$ is a sequence of continuous functions from $[0,1]$, where each $f_n(x)$ is monotone increasing. And suppose that $f_n(x)$ converges to a continuous function $f(x)$ pointwisely on $[0,1]$.

   (a) Show that $f_n(x)$ in fact uniformly converges to $f(x)$ on $[0,1]$.

   (b) Give an example where the uniform convergence fails if the limit function $f(x)$ is not continuous.

2. Suppose that X is a metric space with the distance function $d(\cdot,\cdot)$. For a point $x \in X$ and a subset A, let
$$d(x,A) := \inf\{d(x,y) : y \in A\}.$$

   (a) Let A and B be two disjoint closed subsets in X. Show that
$$f(x) = \frac{d(x,A)}{d(x,A)+d(x,B)} : X \to [0,1]$$
   is continuous.

   (b) Use (a) to verify that, for a closed subset A and an open subset U such that $A \subset U$, there always exists an open set V such that $A \subset V \subset \overline{V} \subset U$.

3. Let (X,Σ,μ) be a finite measure space (i.e., $\mu(X) < \infty$), $\{E_k\}_{k=1}^n$ a collection of measurable sets, and $\{c_k\}_{k=1}^n$ a collection of real numbers. For $E \in \Sigma$, define
$$\nu(E) := \sum_{k=1}^n c_k \, \mu(E \cap E_k).$$

   (a) Show that $\nu \ll \mu$.

   (b) Find the Radon-Nikodym derivative $\frac{d\nu}{d\mu}$ (i.e. find a function g such that $\int f\, d\nu = \int fg\, d\mu$ for all f).

4. (a) Let f and g be absolutely continuous functions on $[0,1]$. Show that their product is also absolutely continuous.

   (b) Give an example of a function on $[0,1]$ which is uniformly continuous but not absolutely continuous.

5. Let $X = C([0,1])$ be the Banach space of all continuous complex-valued functions on $[0,1]$ with the maximum norm. Consider the linear operator $A : X \to X$ defined by
$$(Af)(x) = x \int_0^1 f(y)\, dy, \qquad x \in [0,1].$$

   (a) Show that A is bounded and determine its operator norm.

   (b) Determine the spectrum of the operator A.

6. Let X be a Banach space and $x_0 \in X$ be a nonzero element. Let $X_0 = \text{lin}\,\{x_0\}$ be the linear span by the element x_0. Show that there exists a *closed* linear subspace X_1 of X such that X is the direct sum of X_0 and X_1, i.e., each element $x \in X$ is the unique sum of an element in X_0 and an element in X_1.

7. Let $f = u + iv$ be an entire function such that $|u||v|$ is bounded. Prove that f must be a constant function.

8. Prove that

$$\frac{1}{2\pi}\int_0^{2\pi} e^{\cos\theta}\,d\theta = \sum_{n=0}^{\infty} \frac{1}{(n!2^n)^2}.$$