Fall 2019 - Analysis Prelim Exam- Friday, October 4
University of California Santa Cruz

1. Let f be a continuous odd function defined on $[-1, 1]$. Prove that if

$$\int_{-1}^{1} f(x)x^{2k-1}dx = 0$$

for any $k \in \{1, 2, 3, ...\}$. Then $f \equiv 0$ on $[-1, 1]$.

2. (a) Let X be a locally compact Hausdorff space, $K \subset V \subset X$ where K is compact and V is open. State the Urysohn's Lemma in terms of K and V.

   (b) Let (X, d) be a metric space. For a non-empty subset $A \subset X$, define

$$d_A(x) = \inf\{d(x, a) : a \in A\}\,.$$

   Show that $d_A(x)$ is a uniformly continuous function on X.

   (c) Let A, B be two non-empty closed sets in the metric space (X, d). Construct an explicit continuous function $f : X \to [0, 1]$ using $d_A(x)$ and $d_B(x)$ from (b) such that $f(a) = 0, \forall a \in A$ and $f(b) = 1, \forall\, b \in B$. Explain the relevance of f to the Urysohn's Lemma in (a).

3. Let (X, Σ, μ) be a measure space. Let $\{E_n\}_{n=1}^{\infty}$ be a sequence of μ-measurable sets. Assume that $\sum_{n=1}^{\infty} \mu(E_n) < \infty$. Show that

$$\mu\left(\limsup_{n\to\infty} E_n\right) = 0\,.$$

Recall that

$$\limsup_{n\to\infty} E_n = \cap_{k=1}^{\infty} \cup_{n=k}^{\infty} E_n\,.$$

4. Let (X, Σ, μ) be a finite measure space (i.e., $\mu(X) < \infty$). Consider measures ν and η on X defined by

$$\nu(E) = \int_E f d\mu \quad \text{and} \quad \eta(E) = \int_E g d\mu$$

respectively, where $E \in \Sigma$, the density functions $f(x) > 0, g(x) > 0$ for all $x \in X$. Is $\nu \ll \eta$? If it is, determine the Radon-Nikodym derivative $\frac{d\nu}{d\eta}$. Is $\eta \ll \nu$?

5. Show that the spectrum $\sigma(A) = \{\lambda \in \mathbf{C} : A - \lambda I \text{ is not invertible}\}$ of a bounded linear operator A on a complex Banach space is a *non-empty* compact subset of $\mathbf{C}$. Does the same hold for operators on real Banach spaces ?

6. (a) Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of bounded linear operators on a Banach space X such that $A_n x$ converges for every $x \in X$. Show that $Ax := \lim_{n\to\infty} A_n x$ defines a bounded linear operator A on X.

   (b) Can the same conclusion be drawn if we consider a sequence of bounded linear operators on a normed space ?

7. Let $\Omega \subset \mathbf{C}$ be a connected domain and let $f(z)$ be holomorphic on Ω. Show that neither $\text{Re}[f(z)]$ nor $|f(z)|$ can attain a maximum on Ω unless f is constant.

8. Evaluate the integral

$$I = \int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx.$$