GEOMETRY AND TOPOLOGY PRELIMINARY EXAM
FALL 2021

Problem [1]:

(a) For real constants c_0, c_1, c_2, c_3, consider the function $f : S^3 \to \mathbb{R}$ defined by

$$f(x_0, x_1, x_2, x_3) := c_0 x_0^2 + c_1 x_1^2 + c_2 x_2^2 + c_3 x_3^2.$$

This function descends to a function $\overline{f} : \mathbb{R}P^3 \to \mathbb{R}$.

(i) Suppose $0 < c_0 < c_1 < c_2 < c_3$. Determine the critical points of $\overline{f}$ as well as their critical values.

(ii) Suppose $0 < c_0 = c_1 < c_2 < c_3$. Determine the critical points of $\overline{f}$ as well as their critical values.

(b) Let SO(3) be the 3-dimensional rotation group with determinant 1. Let $T : \mathrm{SO}(3) \to \mathbb{R}$ denote the trace map $T(A) := \mathrm{Trace}(A)$. Determine the critical points of T as well as their critical values.

Problem [2]: On $\mathbb{R}^3$, consider the 1-form $\theta := dz - y\,dx$. Let H be the 2-dimensional distribution given by the kernel of θ: $H := \mathrm{Ker}\,\theta = \{v \in T\mathbb{R}^3 \mid \theta(v) = 0\}$.

(a) Show that there are no 2-dimensional submanifolds of $\mathbb{R}^3$ tangent to H.

(b) Choose vector fields X and Y spanning H at each point of $\mathbb{R}^3$. Compute the flows generated by the vector fields X and Y.

(c) For any smooth function $f \in C^\infty(\mathbb{R}^3)$, consider the vector field W given by

$$W := f\frac{\partial}{\partial z} - B,$$

where B is the unique vector field in H such that

$$\iota_B(d\theta)|_H = df|_H,$$

where ι_B denotes the interior product by B. Let $\phi_W(t)$ be the flow generated by W. Show that this flow preserves H, that is, $\phi_W(t)_*(H) = H$, whenever the flow is defined.

Problem [3]:

(a) Let X be a smooth vector field on a smooth manifold M. Prove or provide a counter-example: If β is a closed form on M, then the Lie derivative $\mathcal{L}_X\beta$ of β is exact.

(b) Find the Lie derivative of the two-form $\beta(x, y) = (x^k + y^k)dx \wedge dy$ on $\mathbb{R}^2$ with respect to the vector field $X(x, y) = x^n\partial_y$ for $k, n \in \mathbb{N}$.

Problem [4]: The n-torus $\mathbb{T}^n$ is a manifold as well as an abelian group.

(a) Describe the translation vector fields on $\mathbb{T}^n$ in terms of coordinates on $\mathbb{T}^n$.

(b) Show that if a vector field on $\mathbb{T}^n$ commutes with all translation vector fields then it is a translation vector field.

Problem [5]: Describe **two** topologically distinct smooth compactifications of $\mathbb{R}^n$. By a *smooth compactification* of $\mathbb{R}^n$ we will mean a smooth closed (= compact without boundary) n-manifold which contains $\mathbb{R}^n$ (or a diffeomorphic copy thereof) as an open dense submanifold. Be sure to justify your answer.

Problem [6]: The alternating group A_5 on five symbols has the following presentation:

$$A_5 = \langle a, b \mid a^5 = b^2 = (ab)^3 = 1\rangle.$$

(This is not obvious.)

(a) Construct a topological space X whose fundamental group is isomorphic to A_5. Explain your construction carefully.

(b) Compute the first homology group $H_1(X)$.

Problem [7]: Let $(G, *)$ be a topological group with multiplication $*: G \times G \to G$. Here are two ways to multiply loops in G based at the identity element $e \in G$:

- Use the usual concatenation of loops:

$$(f \cdot g)(s) := \begin{cases} f(2s), & 0 \le s \le 1/2 \\ g(2s-1), & 1/2 \le s \le 1. \end{cases}$$

- Use the group structure in G:

$$(f * g)(s) := f(s) * g(s), \qquad 0 \le s \le 1.$$

(a) Show that for all loops f and g based at e, we have a homotopy of loops $f \cdot g \simeq f * g$. Thus, the above two ways of multiplying loops define the same product structure on $\pi_1(G, e)$.
(b) Show that $\pi_1(G, e)$ is in fact abelian regardless of whether or not G is abelian.
(c) Compute $\pi_1(\mathbb{T}^n)$ where $\mathbb{T}^n$ denotes the n-torus. Be sure to justify your answer.