Geometry and Topology Prelim, Fall 2022

1. (a) Show that the map $\pi : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}P^n$ where $\pi(x_0, \ldots, x_n) = [x_0 : \ldots : x_n]$ is a submersion.
   (b) Let $f : M \to N$ be a smooth map between smooth manifolds. Show that if f is a submersion, then $df : TM \to TN$ is also a submersion.

2. Consider the distribution $\mathcal{V} = \mathrm{span}(X, Y)$ on $\mathbb{R}^3$ where

$$X = x\frac{\partial}{\partial x} - z\frac{\partial}{\partial z}$$
$$Y = x\frac{\partial}{\partial y} + \frac{\partial}{\partial z}$$

   (a) Is $\mathcal{V}$ involutive?
   (b) Given an arbitrary point $(x, y, z) \in \mathbb{R}^3$, is there an integral submanifold of $\mathcal{V}$ containing (x, y, z)?
   (c) Is there a global coordinate chart with respect to which $X = \partial_1$ and $Y = \partial_2$? In other words, does there exist a diffeomorphism $\phi : \mathbb{R}^3 \to \mathbb{R}^3$, where

$$\phi(x, y, z) = (u(x, y, z), v(x, y, z), w(x, y, z)),$$

   such that $X = d\phi^{-1}(\frac{\partial}{\partial u})$ and $Y = d\phi^{-1}(\frac{\partial}{\partial v})$?

3. Let $S \in GL(n, \mathbb{R})$ be anti-symmetric, i.e. $S^T = -S$.
   (a) Show that

$$G := \{A \in GL(n, \mathbb{R}) : A^T SA = S\}$$

   is a Lie group, with the usual matrix multiplication as the group multiplication. (In other words, show that G is both a smooth manifold and a group, and that multiplication and inversion are smooth maps.)
   (b) Determine the tangent space of G at the identity.

4. Let $\Omega^*(\mathbb{R}^{2n})$ be the de Rham algebra of $\mathbb{R}^{2n}$. For each p such that $1 \leq p \leq n$, find an example of a closed nonzero 2-form $\alpha_p \in \Omega^2(\mathbb{R}^{2n})$ such that the p-fold wedge product with itself is nonzero, but $(p + 1)$-fold wedge product with itself is everywhere zero, that is, $(\alpha_p)^p \neq 0$ but $(\alpha_p)^{p+1} = 0$.

5. The suspension SX of a topological space X is the quotient of $X \times [0, 1]$ by the equivalence relations

$$(x_1, 0) \approx (x_2, 0) \quad \text{and} \quad (x_1, 1) \approx (x_2, 1) \quad \text{for all } x_1, x_2 \in X,$$

   with the quotient topology. I.e., SX is constructed by taking a cylinder over X and then collapsing $X \times \{0\}$ to a point p_0 and $X \times \{1\}$ to a point p_1.

   Express the reduced homology groups $\widetilde{H}_n(SX)$ for all $n \geq 1$ in terms of the reduced homology groups of X.

6. Let X be the space obtained by deleting the points $P = (1, 0)$ and $Q = (-1, 0)$ from $\mathbb{R}^2$.

   (a) The x-axis minus $\{P, Q\}$ is divided into three open intervals, $L_1 = (-\infty, -1), L_2 = (-1, 1)$ and $L_3 = (1, \infty)$, all oriented in the positive direction. Let α be a closed 1-form with compact support on X. Show the following identity of integrals.

$$\int_{L_1} \alpha + \int_{L_2} \alpha + \int_{L_3} \alpha = 0.$$

   (b) Let L_θ be a straight ray from the origin having angle θ from the positive x-axis. Let α be a closed 1-form with compact support in X. Show that the integral of α along the ray L_θ is independent of the angle θ.

   (c) Show that $H^1_c(X) \cong \mathbb{R} \oplus \mathbb{R}$ and exhibit generators of the cohomology classes. Describe this isomorphism.

7. Compute the fundamental group of

$$X := \{(z, w) \in \mathbb{C}^2 : |z| = |w| = 1\} \cup \{(1, w) : w \in \mathbb{C}\}.$$