Geometry and Topology Prelim, Winter 2020

1. Let $F\colon \mathbb{C} \to \mathbb{C}$ be a holomorphic function. Show that F is necessarily orientation preserving at its regular points, i.e., $F^*dx \wedge dy = f\,dx \wedge dy$ with $f \geq 0$.

2. Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a smooth function with $df \neq 0$ at the origin $0 \in \mathbb{R}^n$ and $f(0) = 0$. Show that on a sufficiently small neighborhood of the origin there exists a coordinate system $(x_1, \ldots, x_n)$ such that $f(x_1, \ldots, x_n) = x_1$.

3. Consider the vector fields

$$v_k = x^k \partial_x + y^k \partial_y$$

on $\mathbb{R}^2$, where $k \in \mathbb{Z}_+$.

(a) Show that $[v_k, v_\ell] = (\ell - k)v_{k+\ell-1}$.

(b) Find the (local) flow of v_k. For which k is the vector field v_k complete?

(c) Let φ^t be the flow of v_1. Find the push-forward $\varphi^t_*(v_k)$ and the pull-back $(\varphi^t)^* dx \wedge dy$.

4. Prove that a connected manifold is path-connected.

5. Let

$$A = \{(x, 0, 0) : x \geq 0\} \cup \{(0, y, 0) : y \geq 0\} \cup \{(0, 0, z) : z \geq 0\}$$

denote the union of the non-negative coordinate axes in $\mathbb{R}^3$ and let $M = \mathbb{R}^3 - A$. Compute the fundamental group of M.

6. Let $M := \{(x, y, z) \in \mathbb{R}^3 : z = x^2 - y^2\}$.

(a) Show that M is a submanifold of $\mathbb{R}^3$.

(b) Find smooth vector fields X_1 and X_2 on M such that for every $p \in M$, $\{X_1(p), X_2(p)\}$ is an orthonormal basis for T_pM.

(c) Compute the Gaussian curvature of M, with respect to the Riemannian metric on M induced by the Euclidean metric on $\mathbb{R}^3$, at the origin.

7. Let $L : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be an orthogonal transformation and let $f\colon S^n \to S^n$ denote the restriction of L to the n-sphere. Show that $\deg(f) = \det(f)$.