Geometry and Topology Preliminary Exam. Math. UC Santa Cruz. Spring 2020.

1. Let the "half-rank" of a non-vanishing one-form α at a point p be the maximum integer ℓ such that $\alpha(p) \wedge (d\alpha(p))^\ell \neq 0$

a) Compute the half-rank of $\alpha = dx_5 + x_4 dx_3 + x_2 dx_1$

b) Show that if α is a non-vanishing one-form on a manifold of odd dimension $n = 2k+1$ then its rank at any point is at most k

c) Construct a one-form on $\mathbb{R}^5$ whose half-rank is 1 for points p lying on a linear hyperplane and 2 for all points off this hyperplane.

2. Let K be an invertible symmetric real $n \times n$ matrix. Consider the set of all real $n \times n$ matrices satisfying the equation

$$AKA^t = K$$

a) Show that this set is an embedded submanifold of the space of all real $n \times n$ matrices

b) Compute its dimension

c) Desribe its tangent space at $A = I$ as a linear subspace of the space of all real $n \times n$ matrices.

3. Consider two C^∞-maps $f_1, f_2 \colon \mathbb{R} \to \mathbb{R}^2$, Show that for a full measure set of points $v \in \mathbb{R}^2$ the system of equations $f_1(x) = f_2(x) + v$ has no solutions. Is this still true when the maps are assumed only to be continuous?

4. Problem: Let D be the distribution on $\mathbb{R}^3$ spanned by the following vector fields.

$$X = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}, \qquad Y = z\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + (x+yz)\frac{\partial}{\partial z}.$$

(1) Show that the distribution D is involutive.

(2) Find two commuting vector fields V, W spanning D.

(3) Find a coordinate chart (u, v, w) such that $V = \partial/\partial v$ and $W = \partial/\partial w$.

5. Let $X \subset \mathbb{R}^3$ be constructed from the unit ball by deleting the open ball of radius $1/2$ centered at the origin and adding back in a diameter for this ball. Compute $H_1(X, \mathbb{Z})$ and $H_2(X, \mathbb{Z})$.

6. Evaluate the integral

$$\int_S \frac{x\, dy \wedge dz}{(x^2+y^2+z^2)^{3/2}},$$

where S is the sphere of radius 2 centered at the origin.

7. Let $n \geq 0$. Compute the fundamental group $\pi_1(\mathbb{R}P^n)$ of real projective space $\mathbb{R}P^n$.