GEOMETRY AND TOPOLOGY PRELIMINARY EXAM
SPRING 2021

[1] Consider a real-valued function $f(x, y, z) = (2 - (x^2 + y^2)^{\frac{1}{2}})^2 + z^2$ on $\mathbb{R}^3 - \{z\text{-axis}\}$.

(a) Determine the set of critical points.
(b) Show that 1 is a regular value of f and identify the manifold $M = f^{-1}(1)$.
(c) Let N be a surface in $\mathbb{R}^3$ given by $x^2 + y^2 = 4$. Show that M and N are transverse, and identify their intersection $M \cap N$.

[2] Let $Y = \mathbb{R}^3 - \{z\text{-axis}\}$. A smooth closed one-form α on Y has the property that its integral around the unit circle in the xy-plane, oriented counterclockwise, is 1. Write down an expression for one such α in Cartesian coordinates.

[3] Find all connected 3-dimensional Lie groups G with the property that whenever X is a nonzero element of the Lie algebra of G we have that the curve $t \mapsto \exp(tX)$ is a topological circle in G

[4] Suppose M is a compact connected simply connected 4-manifold without boundary.

(a) Explain why M is orientable.
(b) Compute the homology groups $H_0(M; \mathbb{Z})$, $H_1(M; \mathbb{Z})$, $H_3(M; \mathbb{Z})$, and $H_4(M; \mathbb{Z})$.
(c) Explain why $H_2(M; \mathbb{Z})$ is a finitely generated free abelian group.

[5] Construct a two-dimensional CW-complex X whose fundamental group $\pi_1(X)$ is isomorphic to $\mathbb{Z}/5\mathbb{Z}$. In addition, construct a three-dimensional CW-complex with the same property. Be sure to justify your answer.

[6] For each k-tuple of vector fields on $\mathbb{R}^3$ shown below, either find smooth coordinates (s_1, s_2, s_3) in a neighborhood of $(1, 1, 0)$ such that $V_i = \partial/\partial s_i$ for $i = 1, \cdots, k$, or explain why there are none.

$$(a) \quad k = 2, \qquad V_1 = x\frac{\partial}{\partial x} - (y+2)\frac{\partial}{\partial y}, \quad V_2 = x\frac{\partial}{\partial x} + (y+2)\frac{\partial}{\partial y}.$$

$$(b) \quad k = 3, \quad V_1 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}, \quad V_2 = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, \quad V_3 = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}.$$

[7] Let e_1, e_2, e_3, e_4 be the standard basis for $\mathbb{R}^4$. For two linearly independent vectors $u, v \in \mathbb{R}^4$, let

$$u \wedge v = \sum_{i<j} W_{ij}\, e_i \wedge e_j.$$

(a) Show that this defines a well-defined map

$$f : G_2(\mathbb{R}^4) \to \mathbb{R}P^5$$

from the Grassmannian $G_2(\mathbb{R}^4)$ of 2-dimensional subspaces in $\mathbb{R}^4$ to the projective space $\mathbb{P}(\bigwedge^2 \mathbb{R}^4) \cong \mathbb{R}P^5$. Namely, for each 2-dimensional subspace W, choose a basis u, v, and let

$$f(W) := [W_{12} : W_{13} : \cdots : W_{34}].$$

(b) Show that this map is a smooth embedding.
(c) Show that the image of this embedding is precisely given by the submanifold of $\mathbb{R}P^5$ defined by a homogeneous equation

$$W_{12}W_{34} - W_{13}W_{24} + W_{14}W_{23} = 0.$$