GEOMETRY AND TOPOLOGY PRELIMINARY EXAM

Spring 2022

[1] Consider the set of the equations in $\mathbb{R}^3$.

$$\begin{cases} x^3 + y^3 + z^3 = 1 \\ z = xy \end{cases}$$

Prove or disprove that its solution set is a submanifold in $\mathbb{R}^3$.

[2] Consider the distribution $\mathcal{V} = \text{span}(X, Y)$ on $\mathbb{R}^3$ where

$$X = x\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + x(y+1)\frac{\partial}{\partial z}$$

$$Y = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}$$

(1) Prove that $\mathcal{V}$ is involutive.
(2) Check that $[X - xY, Y] = 0$.
(3) Find a coordinate chart on $\mathbb{R}^3$ such that $\mathcal{V} = \text{span}(\partial_1, \partial_2)$.

[3] (Coordinate Change and Volume Form) Let $S^2(r)$ be the sphere of radius $r > 0$ in $\mathbb{R}^3$, and induce a Riemannian metric on $S^2(r)$ from the standard Euclidean metric on $\mathbb{R}^3$. Let $\iota : S^2(r) \to \mathbb{R}^3$ be the inclusion map. Let v be a unit vector field on $\mathbb{R}^3 - \{0\}$ normal to all of the concentric spheres $S^2(r)$, $r > 0$.

(1) Identify the vector field v.

(2) Show that the Riemannian volume form ω for the sphere $S^2(r)$ in Cartesian coordinate system is given by

$$\omega = \iota^* \left[\frac{1}{r}(x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy) \right],$$

where $r = \sqrt{x^2 + y^2 + z^2}$ is the distance from the origin.

(3) Let (ρ, θ, h) be the cylindrical coordinate system for $\mathbb{R}^3$. (For any point $P \in \mathbb{R}^3$, h is its z-coordinate, ρ is the distance from the z-axis, and θ is the angle in xy-plane measured from the positive x-axis.) Show that in the cylindrical coordinate system, the Riemannian volume form ω for $S^2(r)$ in part (1) can be written as

$$\omega = \iota^* \left(r\,d\theta \wedge dh \right),$$

where $r = \sqrt{\rho^2 + h^2}$ is the distance from the origin.

[4] (Cohomology with Compact Support) Let $X = \mathbb{R}^2 - \{0\}$. Let $Z_c^1(X)$ be the set of closed 1-forms on X with compact support. Let L be a ray from the origin to ∞. Consider a map

$$I = \int_L : Z_c^1(X) \longrightarrow \mathbb{R},$$

defined by the integral on the ray L.

(1) Show that the above map I is independent of the choice of the ray L from the origin.

(2) Let $\varphi(r)$ be a bump function in $r > 0$ with compact support near $r = 1$ and total integral 1 along $r > 0$. Let $\alpha = \varphi(r)dr \in \Omega^1(X)$ in polar coordinates. Show that α generates a nonzero cohomology class with compact support in $H_c^1(X)$.

(3) Show that the above map I induces an isomorphism $I_* : H_c^1(X) \to \mathbb{R}$.

[**5**] (1) Let A_* and B_* be chain complexes of abelian groups. Prove that if two chain maps $f_*, g_* : A_* \to B_*$ are homotopic, then the induced maps on homology $H_n(A_*) \to H_n(B_*)$ coincide for each $n \in \mathbb{Z}$.

(2) Let X and Y be topological spaces with singular homology $H_*(X)$ and $H_*(Y)$. Is every homomorphism of graded abelian groups $H_*(X) \to H_*(Y)$ induced by a continuous map $X \to Y$? If the answer is yes, prove it. If the answer is no, provide an explicit counter-example.

[**6**] Let X be the space obtained by gluing two Möbius bands along their boundary. Here a *Möbius band* is the quotient space $([-1,1] \times [-1,1])/(1,y) \sim (-1,-y)$.

(1) Determine the fundamental group of X.

(2) Is X homotopy equivalent to a compact orientable surface of genus g for some g?