GEOMETRY AND TOPOLOGY PRELIMINARY EXAM
SUMMER 2023

(1) Prove that a smooth submersion $F : M \to N$ between smooth manifolds is an open map, i.e. for every open set $U \subseteq M$, the image $F(U) \subseteq N$ is open.

(2) Consider the following subspace of $\mathbb{R}^2$:
$$W_c = \{(x, y) \in \mathbb{R}^2 \mid (x^2 - 1)(x - c) = y^2\}.$$
Find all the parameters $c \in \mathbb{R}$ for which W_c is an embedded submanifold of $\mathbb{R}^2$.

(3) Let M be a smooth, compact manifold which admits a nowhere vanishing smooth vector field. Prove that there exists a smooth map $F : M \to M$ that is homotopic to the identity and has no fixed points.

(4) Let ω be an n-form on an n-dimensional manifold M. Assume that $\omega_p \neq 0$ at some point $p \in M$. Show that there exist local coordinates $x_1, \ldots, x_n$ near p such that $\omega = dx_1 \wedge \ldots \wedge dx_n$.

(5) Let $L\colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be an orthogonal transformation and let $f\colon S^n \to S^n$ denote the restriction of L to the n-sphere. Show that $\deg(f) = \det(L)$. (Make sure to clearly state what definition of the degree you are using.) When is f orientation preserving/reversing?

(6) Calculate the fundamental group of the topological space X constructed by deleting a circle from $\mathbb{R}^3$.

(7) Prove or provide a counter-example: There is no continuous map $F : S^2 \to S^1$ such that $F \circ \iota : S^1 \to S^1$ is the identity map, where $\iota : S^1 \hookrightarrow S^2$ is the inclusion of S^1 into S^2 as the equator.