GEOMETRY & TOPOLOGY PRELIMINARY EXAM
SPRING 2024

1. Let M, N be smooth manifolds without boundary and let $F : M \to N$ be a smooth map.
(1) Describe the smooth structure of the tangent bundle TM of M. (No need to show the smooth compatibility of charts.)
(2) Describe how the global differential $dF : TM \to TN$ is defined.
(3) Show that $dF : TM \to TN$ is a smooth map.

2. Let $U(n) = \{A \in GL(n, \mathbb{C}) | A^* A = I_n\}$ be the unitary group of degree n, where $A^\star = \bar{A}^T$ is the conjugate transpose of A. Show that $U(n)$ is an embedded Lie subgroup of $GL(n, \mathbb{C})$.

3. Let M be a smooth manifold and let X be a smooth vector field on M. Suppose that X is not complete. Show that there is an integral curve of X that is not contained in any compact subsets of M.

4. Let $E_{\mathbb{C}}$ be a complex line bundle and let $E_{\mathbb{R}}$ be the same line bundle regarded as a real vector bundle. (Thus $\operatorname{rank}_{\mathbb{R}} E_{\mathbb{R}} = 2$.) Prove that $E_{\mathbb{C}}$ is trivial if and only if $E_{\mathbb{R}}$ is trivial.

5. Which of the following manifolds are orientable and which are not: the real projective spaces $\mathbb{RP}^n$ for $n = 1, 2, \ldots$, the special orthogonal groups $\mathrm{SO}(n, \mathbb{R})$ for $n = 1, 2, \ldots$, the complex projective spaces $\mathbb{CP}^n$ for $n = 1, 2, \ldots$. Give a brief justification to your answer. (No detailed proof is required.)

6. Consider the 1-form
$$\omega = \frac{x}{x^2 + y^2} dy - \frac{y}{x^2 + y^2} dx$$
on $\mathbb{R}^2 \setminus (0, 0)$ and the unit circle $\gamma = (\cos(\alpha), \sin(\alpha))$ in $\mathbb{R}^2 \setminus (0, 0)$. Find $\int_\gamma \omega$. Is ω a trivial element in $H^1_{dR}(\mathbb{R}^2 \setminus (0, 0))$? Explain why.

7. What is the universal covering of $S^1 \vee S^2$? Draw it. Find $\pi_1(S^1 \vee S^2)$ and $\pi_2(S^1 \vee S^2)$.