GEOMETRY AND TOPOLOGY PRELIMINARY EXAM

Winter 2021

[1] Let X be the 2-sphere with a whisker connecting the north pole to the south pole. Compute the fundamental group and the homology groups of X.

[2] Consider the real projective space $\mathbb{R}P^4$.

(1) Describe a cell structure of $\mathbb{R}P^4$.
(2) Describe the attaching maps of cells in (1).

[3] Let X be a vector field on $\mathbb{R}P^2$ which is described by

$$X = u\frac{\partial}{\partial u} + 2v\frac{\partial}{\partial v},$$

on the open local coordinate chart $U_0 = \{[1; u, v] \in \mathbb{R}P^2 \mid u, v \in \mathbb{R}\}$. Find the expression of X on other local coordinate charts $U_1 = \{[x; 1; y] \in \mathbb{R}P^2\}$ and $U_2 = \{[s; t; 1] \in \mathbb{R}P^2\}$, and identify points of $\mathbb{R}P^2$ where X vanishes.

[4] On $M = \mathbb{R}^3 - \{z\text{-axis}\}$, consider vector fields X, Y given as follows.

$$X = (x - 2y)\frac{\partial}{\partial x} + (2x + y)\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}, \qquad Y = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}.$$

(1) Show that X, Y define an involutive rank 2 distribution D on M.
(2) Compute and describe all the integral submanifolds of D.

[5] A one-form α on the plane $\mathbb{R}^2$ has the property that $\int_I \alpha = 0$ for all line segments I which have one endpoint at the origin.

(1) Find the general algebraic expression for any such α in terms of x, y, dx and dy
(2) Suppose that in addition $\int_{S^1(r)} \alpha = \pi r$ where $S^1(r)$ denotes the circle centered at the origin with radius r oriented counterclockwise. Now find the general expression for all such α.

[6] Let T^2 be the standard two-torus with angular coordinates θ^1, θ^2.

(1) Describe what is meant by a closed curve c in t^2 having $(2, 3)$ winding. Also draw a picture of such a c on the "flat torus" model of T^2.
(2) Find a closed one-form α having $\int_c \alpha = 0$ and $\int_{e_1} \alpha = 1$ where c is as in (1) and e_1 is a closed curve with $(1, 0)$ winding. Express your form in terms of $d\theta_1$ and $d\theta_2$.
(3) Is your form α unique? If 'yes', prove it. If 'no' describe the ambiguities – i.e. the set of all such α.

[7] A smooth function f on a Riemannian manifold satisfies $\|\nabla f\| = 1$ everywhere

(1) Use the relation induced between df and ∇f by the Riemannian metric to derive an inequality between $df_p(v_p)$, and $\|v_p\|$ valid for all $p \in M$ and all vectors $v_p \in T_pM$. State a necessary and sufficient condition for your inequality to become an equality.
(2) Use (1) to prove that the integral curves of ∇f are geodesics.
(3) When $M = \mathbb{R}^n$ with its Euclidean metric, describe ∇f's integral curves.