GEOMETRY AND TOPOLOGY PRELIMINARY EXAM
WINTER 2022

(1) Let M be a smooth manifold with the tangent bundle $\pi : TM \to M$. Prove that there exists a smooth vector field X on TM (i.e. $X \in \Gamma^\infty(T(TM))$) such that $d_{(p,v)}\pi(X_v) = v$ for every $(p, v) \in TM$ (i.e. $p \in M$ and $v \in T_pM$).

(2) Let $X = y^2 \frac{\partial}{\partial x}$ and $Y = x^2 \frac{\partial}{\partial y}$ be two vector fields on $\mathbb{R}^2$.
   (a) Prove that each of X and Y is complete.
   (b) Prove that the vector field $X + Y$ is not complete. *(Hint: consider the integral curve of $X + Y$ starting at $(a, a) \in \mathbb{R}^2$ for $a \neq 0$.)*

(3) Let D be a distribution in $\mathbb{R}^3$ spanned by the vectors $X = xy\frac{\partial}{\partial x} + \frac{\partial}{\partial z}$ and $Y = \frac{\partial}{\partial y}$.
   (a) Find an integral manifold of D passing through the origin.
   (b) Is D involutive?

(4) (a) Define what it means for a map $f : \mathbb{R} \to \mathbb{R}$ to be analytic.
   (b) True/False. Prove or give a counterexample. "A non-constant analytic map $f : \mathbb{R} \to \mathbb{R}$ is an open mapping. "

(5) Let $K \subset \mathbb{R}^3$ be a smooth 'knot', i.e. a smoothly embedded image of the circle. Let $\nu(K)$ be its normal bundle, so that $\nu(K) \subset K \times \mathbb{R}^3$ consists of pairs (p, v) such that $p \in K, v \in \mathbb{R}^3, v \perp T_pK$. Define the 'exponential map' $exp : \nu(K) \to \mathbb{R}^3$ by $exp(p, v) = p + v$.
   (a) Show that the exponential map is a diffeomorphism onto its image provided that we restrict the normal vectors v to a small enough ball. In other words, show that there is an $\epsilon > 0$ such that the restriction of exp to the set of (p, v) with $\|v\| < \epsilon$ is a diffeomorphism onto its image.
   (b) Show that the exponential map cannot be a global diffeomorphism.

(6) (a) What is the dimension of $\Lambda^2(\mathbb{R}^{4*})$, the space of constant two-forms on $\mathbb{R}^4$?
   (b) Let μ be a non-vanishing constant 4-form on $\mathbb{R}^4$, for example $\mu = dx \wedge dy \wedge du \wedge dv$ where x, y, u, v are standard linear coordinates. Define a map

$$Q : \Lambda^2\mathbb{R}^{4*} \to \mathbb{R}, \text{ by } \omega \wedge \omega = Q(\omega)\mu.$$

   Show that Q is a non-degenerate quadratic form. Compute its index as a quadratic form. Does the index depend on the choice of μ?

(7) Let X denote the 2-dimensional torus with an open disk removed.
   (a) Compute the homology $H_*(X)$.
   (b) The boundary of X is a copy of the circle S^1. Does X retract onto its boundary?

(8) (a) Let $n \geq 1$, $d \in \mathbb{Z}$ and let $f : S^n \to S^n$ be a degree d map on the n-sphere. Let X be the space obtained from S^n by attaching an $(n+1)$-disk via the attaching map f. Compute the homology $H_*(X)$.

(b) Construct a space Y whose reduced homology is as follows:

$$\widetilde{H}_i(Y) = \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z} \oplus \mathbb{Z} & i = 1000 \\ \mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z} & i = 2000 \\ 0 & \text{otherwise} \end{cases}$$