Winter 2020 - Analysis Prelim
January 24, 2020

1. Prove that every compact Hausdorff space X is normal, i.e., for each pair A, B of disjoint closed subsets of X there exist disjoint open sets U, V such that $U \supseteq A$ and $V \supseteq B$ (Compactness here means every open cover has a finite subcover).

2. Let $C[0,1] = \{f : [0,1] \to \mathbb{R} : \ f \text{ is continuous in } [0,1]\}$ be the space of continuous functions with the norm
$$\|f\|_0 = \max\{|f(x)| : x \in [0,1]\}$$
And let $C^1[0,1] = \{f : [0,1] \to \mathbb{R} : \ f \text{ and } f' \text{ are all continuous in } [0,1]\}$ be the space of continuously differentiable functions with the norm
$$\|f\|_1 = \max\{|f(x)| + |f'(x)| : x \in [0,1]\}.$$
Show that a bounded set in $C^1[0,1]$ is pre-compact in $C[0,1]$.

3. Let (X, Λ, μ) be a finite measure space, and let $f : X \to \mathbb{R} \cup \{\infty\}$ is finite μ-a.e. and measurable. Show that, for any $\epsilon > 0$, there exists $M > 0$ such that $|f(x)| \leq M$ except on a set of measure less than ϵ.

4. Let g be an integrable function on a set E. And suppose $\{f_n\}$ is a sequence of measurable functions such that $|f_n| \leq g, \forall n = 1, 2, \dots$ a.e. on E. Show that
$$\int_{\mathrm{E}} \underline{\lim}_n f_n d\mu \leq \underline{\lim}_n \int_{\mathrm{E}} f_n d\mu \leq \overline{\lim}_n \int_{\mathrm{E}} f_n d\mu \leq \int_{\mathrm{E}} \overline{\lim}_n f_n d\mu.$$

5. Let A be a bounded linear operator on a Banach space X. Denote by $R(A) = \{Ax : x \in X\}$ and $N(A) = \{x \in X : Ax = 0\}$ the range and the nullspace of A. Assume that $R(A)$ is dense in X. Show that the following are equivalent:

   (i) there exists a bounded linear operator B on X such that $AB = BA = I_X$;
   (ii) there exists a constant $\gamma > 0$ such that $\|x\| \leq \gamma \|Ax\|$ for all $x \in X$;
   (iii) $N(A) = \{0\}$ and $R(A)$ is closed.

6. Let $T : X \to Y$ be a linear operator between normed spaces. Show that T is bounded if and only if T is continuous.

7. For a real number a, evaluate the following integral using residue theory,
$$I = \int_{-\infty}^{\infty} \frac{e^{iax}}{(1+x^2)^2}\, dx.$$

8. Show that every non-constant polynomial $p(z)$ has at least one complex zero $z_0 \in \mathbb{C}$.