1. Let \( G = GL(n, \mathbb{C}) \) be the group of all \( n \times n \) invertible matrices with complex entries. Show that \( G \) contains a subgroup \( B \) with the following properties:
   (a) \( B \) is solvable.
   (b) \( G = \cup_{g \in G} gBg^{-1} \), i.e. \( G \) is the union of the conjugates of \( B \).
(Hint: consider the upper triangular matrices in \( G \).)

2. Let \( G \) be a finite group of order \( pqr \) where \( p, q, r \) are distinct primes. Prove that \( G \) is solvable.

3. What is the automorphism group of \( \mathbb{Z}_2 \times \mathbb{Z}_4 \) isomorphic to? Justify your answer.

4. Let \( A \) be a \( n \times n \) complex matrix such that \( A^k \) is the identity matrix for some \( k > 0 \). Prove that \( A \) is diagonalizable.

5. Let \( A = J_6(\lambda) \) be a Jordan block of \( 6 \times 6 \) matrix with eigenvalue \( \lambda \).
   (a) Suppose \( \lambda = 0 \). Find the Jordan canonical forms of \( A^2 \) and \( A^3 \).
   (b) Suppose \( \lambda \neq 0 \). Find the Jordan canonical forms for \( A^2 \) and \( A^3 \).

6. Let \( A \) be a real symmetric and positive definite matrix. Prove that the maximal matrix entries are on the diagonal.

7. Let \( I = (2, x) \) be the ideal generated by 2 and \( x \) in the ring \( R = \mathbb{Z}[x] \). Show that the element \( 2 \otimes 2 + x \otimes x \in I \otimes_R I \) cannot be written as \( a \otimes b \) for some \( a, b \in I \).

8. (a) Show \( (x^d - 1)|(x^n - 1) \) if and only if \( d|n \).
   (b) Let \( \ell \) be a prime and let \( \zeta_\ell \) be a primirive \( \ell \)th root of unity over \( \mathbb{F}_p \) for a prime \( p \). Show that \( \zeta_\ell \in \mathbb{F}_{p^n} \) if and only if \( \ell|p^n - 1 \).
   (c) Find the splitting field of \( x^7 - 1 \in \mathbb{F}_{11}[x] \).

9. Let \( \alpha = \sqrt{2 + \sqrt{2}} \). Show that \( \mathbb{Q}(\alpha) \) is a cyclic Galois extension of degree 4.