Algebra Preliminary Exam, Fall 2012

1. Show that all groups of order 36 are solvable.

2. Let $H$ be a proper subgroup of a finite group $G$. Show that there exists an element $g \in G$ which is not conjugate to any element of $H$. (Hint: bound the number of elements in $\bigcup_{g \in G} gHg^{-1}$.)

3. Let $F := \mathbb{Z}/2\mathbb{Z}$.
   (a) Show that $X^2 + X + 1$ is the only irreducible polynomial of degree 2 in $F[X]$.
   (b) Show that $X^4 + X^3 + 1$ is irreducible in $F[X]$.
   (c) Show that $X^4 + X^3 + 1$ is irreducible in $\mathbb{Q}[X]$.

4. Let $(V, \langle \cdot, \cdot \rangle)$ be a finite dimensional complex inner product space. Recall that an operator $T$ on $V$ is said to be unitary if for every pair $(\mathbf{u}, \mathbf{v})$ of vectors from $V$, $\langle T(\mathbf{u}), T(\mathbf{v}) \rangle = \langle \mathbf{u}, \mathbf{v} \rangle$. Prove that $T$ is unitary if and only if the following statement holds: There exists an orthonormal basis $\mathcal{B} = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$ consisting of eigenvectors of $T$ and every eigenvalue of $T$ has norm 1.

5. Let $T$ be a finite dimensional vector space and $T$ an operator on $V$.
   (a) Assume $V = \langle T, \mathbf{u} \rangle \oplus \langle T, \mathbf{w} \rangle$ where $\mu_{T,\mathbf{u}}(x) = \mu_{T,\mathbf{w}}(x) = p(x)$ is irreducible. Prove there is a $T$-invariant subspace $X$ such that $U \cap X = W \cap X = \{0\}$.
   (b) Let $V = U \oplus W$ where $U$ and $W$ are $T$-invariant. Set $T_U = T|U$, $T_W = T|W$, $f(x) = \mu_{T_U}(x)$, $g(x) = \mu_{T_W}(x)$. Assume for every $T$-invariant subspace $X$ that $X = (X \cap U) + (X \cap W)$. Prove that $f(x)$ and $g(x)$ are relatively prime.

6. Let $\text{GL}_n(\mathbb{R})$ denote the group of $n \times n$ non-singular matrices with real coefficients and $\text{Sym}_n(\mathbb{R})$ the space of $n \times n$ symmetric matrices. Define an action of $\text{GL}_n(\mathbb{R})$ on $\text{Sym}_n(\mathbb{R})$ by $Q \circ A = Q^{tr}AQ$. Determine, with a proof, the number of orbits for this action.

7. Let $k$ be a field and let $A = k[x, y]$. Consider the maximal ideal $m = (x, y)$. Is $m$ a projective $A$-module? Is $m$ a flat $A$-module?
8. Let $F$ be a field. Let $f(x) \in F[x]$ be the characteristic polynomial of some matrix with coefficients in $F$. Show that all matrices with coefficients in $F$ with characteristic polynomial $f(x)$ are similar over $F$ if and only if the factorization of $f(x)$ into irreducibles in $F[x]$ has no repeated roots.

9. Let $p$ be an odd prime. Find, with proof, the Galois group of $x^p - 2$ over $\mathbb{Q}$. (Hint: your answer will be a semi-direct product of two cyclic groups.)