Questions in Algebra for Preliminary Exam (Fall 2015)

1. Let $G$ be a group of order $p(p + 1)$, where $p$ is a prime. Prove that $G$ has a normal subgroup of either order $p$ or $p + 1$.

2. Let $R$ be a commutative ring with 1 and $a \in R$. Prove that $a$ is an element of every maximal ideal of $R$ iff $1 - ab$ is a unit for all $b \in R$.

3. Let $D = \mathbb{Z}[\sqrt{5}] = \{m + n\sqrt{5} | m, n \in \mathbb{Z}\}$ — a subring of the field of real numbers and necessarily an integral domain — and $F = \mathbb{Q}[\sqrt{5}]$ its field of fractions. Show the following:
   (a) $x^2 + x - 1$ is irreducible in $D[x]$ but not in $F[x]$.
   (b) $D$ is not a unique factorization domain.

4. Consider the complex matrix
   \[ A = \begin{pmatrix} 1 & -1 \\ 2 & 2 \end{pmatrix}. \]
   Find a unitary matrix $U$ and a positive definite hermitian matrix $P$ such that $A = UP$.

5. Let $p$ be an odd prime number, and consider the set $X$ of $3 \times 3$ matrices with entries in the finite field $\mathbb{F}_p$ whose characteristic polynomial is $T^3 - T$.
   (a) Show that the conjugation-action of $GL_3(\mathbb{F}_p)$ on $X$ is transitive.
   (b) How many elements are there in $X$?

6. Let $A$ be a complex $3 \times 3$ matrix.
   (a) Suppose that $A$ has eigenvalues $\lambda_1, \lambda_2, \lambda_3$. What are the eigenvalues of the second exterior (= alternating) power $\wedge^2 A$?
   (b) Suppose that the linear operators $A, \wedge^2 A$ and $\wedge^3 A$ have trace 2, $-1$ and $-2$, respectively. Prove that $A$ is diagonalisable.
7. Let $p$ be a prime number and $r \geq 1$ an integer not divisible by $p$. Consider the field $V = \mathbb{F}_p^r$ as a vector space over the field $\mathbb{F}_p$. Let $\sigma : V \to V$ be the ($\mathbb{F}_p$-linear) Frobenius map, $\sigma(x) = x^p$.

Find the Jordan canonical form for $\sigma$ (after extending the ground field to $\mathbb{F}_p$). (Hint. You may want to apply the Normal Basis Theorem, which says, that for any finite Galois extension $K/F$ of fields with group $G$, there exists $x \in K$ such that the $G$-orbit $\{g.x\}_{g \in G}$ forms an $F$-basis of $K$.)

8.

(a) Find the Galois group of the polynomial $x^4 - 3$ over $\mathbb{Q}$.

(b) Find the Galois group of the polynomial $x^4 - 3$ over $\mathbb{F}_5$.

9. True or False (give a proof or counterexample): Every flat finitely generated $\mathbb{Z}/6\mathbb{Z}$-module is free.