Questions in Algebra for Preliminary Exam (Fall 2015)

1. Let G be a group of order $p(p+1)$, where p is a prime. Prove that G has a normal subgroup of either order p or $p+1$.

2. Let R be a commutative ring with 1 and $a \in R$. Prove that a is an element of every maximal ideal of R iff $1 - ab$ is a unit for all $b \in R$.

3. Let $D = \mathbb{Z}[\sqrt{5}] = \{m + n\sqrt{5} | m, n \in \mathbb{Z}\}$ — a subring of the field of real numbers and necessarily an integral domain — and $F = \mathbb{Q}[\sqrt{5}]$ its field of fractions. Show the following:
(a) $x^2 + x - 1$ is irreducible in $D[x]$ but not in $F[x]$.
(b) D is not a unique factorization domain.

4. Consider the complex matrix

$$A = \begin{pmatrix} 1 & -1 \\ 2 & 2 \end{pmatrix}.$$

Find a unitary matrix U and a positive definite hermitian matrix P such that $A = UP$.

5. Let p be an odd prime number, and consider the set X of 3×3 matrices with entries in the finite field $\mathbb{F}_p$ whose characteristic polynomial is

$$T^3 - T.$$

(a) Show that the conjugation-action of $GL_3(\mathbb{F}_p)$ on X is transitive.

(b) How many elements are there in X?

6. Let A be a complex 3×3 matrix.

(a) Suppose that A has eigenvalues $\lambda_1, \lambda_2, \lambda_3$. What are the eigenvalues of the second exterior (= alternating) power $\wedge^2 A$?

(b) Suppose that the linear operators A, $\wedge^2 A$ and $\wedge^3 A$ have trace 2, -1 and -2, respectively. Prove that A is diagonalisable.

7. Let p be a prime number and $r \geq 1$ an integer not divisible by p. Consider the field $V = \mathbb{F}_{p^r}$ as a vector space over the field $\mathbb{F}_p$. Let $\sigma : V \to V$ be the ($\mathbb{F}_p$-linear) Frobenius map, $\sigma(x) = x^p$.

Find the Jordan canonical form for σ (after extending the ground field to $\overline{\mathbb{F}}_p$). (*Hint.* You may want to apply the Normal Basis Theorem, which says, that for any finite Galois extension K/F of fields with group G, there exists $x \in K$ such that the G-orbit $\{g.x\}_{g \in G}$ forms an F-basis of K.)

8.

(a) Find the Galois group of the polynomial $x^4 - 3$ over $\mathbb{Q}$.

(b) Find the Galois group of the polynomial $x^4 - 3$ over $\mathbb{F}_5$.

9. True or False (give a proof or counterexample): Every flat finitely generated $\mathbb{Z}/6\mathbb{Z}$-module is free.