Algebra Prelim, Fall 2016

1. Show that every group of order 2016 has a subgroup of order 28.

2. Determine the minimal $n \in \mathbb{N}$ such that the quaternion group Q_8 of order 8 is isomorphic to a subgroup of the symmetric group $\mathrm{Sym}(n)$. (Hint: Consider the resulting action of Q_8 on $\{1, 2, \dots, n\}$ and use that every non-trivial subgroup of Q_8 contains the unique element of order 2.)

3. Let p be a prime. Consider the subring R of $\mathbb{Q}$ consisting of all fractions a/b with $a, b \in \mathbb{Z}$ such that b is not divisible by p. Show that every non-zero ideal of R is of the form $p^k R$ for some integer $k \geq 0$.

4. Suppose that A is a 5×5 complex matrix satisfying

$$A^4 + 4A^2 = 4A^3.$$

(a) What are the possible eigenvalues of A?

(b) Determine all possible Jordan canonical forms of A.

5. Prove or disprove: For any integer $n \geq 1$, any $n \times n$ matrix B over an algebraically closed field k is diagonalisable, if $B^r = I_n$ for some integer $r \geq 1$.

6. Prove or disprove: Any vector $\vec{v} \in \mathbb{R}^3$ is the cross product $\vec{u} \times \vec{w}$ for some vectors $\vec{u}, \vec{w}$ in $\mathbb{R}^3$.

7. Let R be a commutative ring with identity. For each of the following, either prove the statement or give a counterexample:

(a) The tensor product of two injective R-modules is injective.

(b) The tensor product of two flat R-modules is flat.

(Note: you should prove or disprove these statements from scratch, not apply theorems that render the conclusions trivial.)

8. Let $\mathbb{C}(x)$ denote the field of rational functions over $\mathbb{C}$ in an indeterminate x. For each $a \in \mathbb{C}$, let

$$\sigma_a \colon \mathbb{C}(x) \longrightarrow \mathbb{C}(x)$$

denote the field automorphism that substitutes x by $x + a$. Let G denote the group of automorphisms $\{\sigma_a : a \in \mathbb{C}\}$. Determine with proof the subfield $\mathbb{C}(x)^G$, the subfield of $\mathbb{C}(x)$ fixed by G.

9. Let λ be an eigenvalue of a linear transformation T of a finite dimensional vector space V over an algebraically closed field F. Let $r_k = \dim_F (T - \lambda)^k V$, i.e. r_k is the rank of the linear transformation $(T - \lambda)^k$ on V. Prove that for any $k \geq 1$, the number of Jordan blocks of T corresponding to λ of size k is $r_{k-1} - 2r_k + r_{k+1}$.