Algebra Preliminary Exam, Fall 2017

1. Show that a solvable group G cannot have subgroups $N \triangleleft H \leq G$ such that H/N is a non-abelian simple group.

2. Let G be a finite group of order n and let k denote the number of conjugacy classes of elements of G. Show that the number of pairs (x, y) with $x, y \in G$ such that $xy = yx$ is equal to nk.

3. Let $f\colon R \to S$ be a homomorphism between two commutative rings R and S.

(a) Show that if P is a prime ideal of S then $f^{-1}(P)$ is a prime ideal of R.

(b) Prove or disprove by a counterexample: If M is a maximal ideal of S then $f^{-1}(M)$ is a maximal ideal of R.

4. (a) Determine the Gram matrix of the bilinear form $(A, B) = \operatorname{trace}(AB)$ on $M_n(\mathbb{R})$ with respect to the standard basis $\{E_{ij}\}$.

(b) Determine an orthogonal basis for this form.

(c) Determine the signature of the form.

5. Let A and B be $n \times n$-matrices over a field. Prove that AB and BA have the same characteristic polynomial. Give an example where AB and BA do not have the same minimal polynomial.

6. Let A be a square complex matrix.

(a) Prove or disprove that A is diagonalizable if and only if its minimal polynomial has only simple roots.

(b) Prove or disprove that if A^k is the identity matrix for some positive integer k then A is diagonalizable.

7. Let R be a ring and let M be a left R-module. Show that if M is projective then M is flat.

8. Let V_i, $i = 1, 2, 3$, be 4-dimensional vector spaces over a field F and let $T_i\colon V_i \to V_i$ be F-linear maps, for $i = 1, 2, 3$, with the same minimal polynomial $(x - \lambda)^2(x - \mu) \in F[x]$, where $\lambda \neq \mu$ are elements in F. For $i = 1, 2, 3$, consider V_i as $F[x]$-module in via $x \cdot v_i = T_i(v_i)$, for $i = 1, 2, 3$, and $v_i \in V_i$. Show that two of the three $F[x]$-modules V_1, V_2, V_3 must be isomorphic.

9. Let F be a finite field with q elements.

(a) Show that q is a power of a prime p.

(b) Show that if $x \in F$ has the property $x^p = 1$ then $x = 1$.

(c) Show that if two non-zero elements of F have the same multiplicative order then each of them is a power of the other.