Algebra Preliminary Exam, Fall 2018

1. Show that every group of order 200 is solvable.

2. Let G be a finite group, p a prime, and suppose that G has an abelian Sylow p-subgroup P. Assume that X and Y are subsets of P and $g \in G$ with $Y = gXg^{-1}$.

(a) Show that both P and gPg^{-1} are Sylow p-subgroups of $N_G(Y)$.

(b) Show that there exists $h \in N_G(P)$ such that $Y = hXh^{-1}$.

3. Let K be a field and let D be a finite-dimensional K-algebra with no zero-divisors except 0. Show that D is a division ring.

4. Let m and n be positive integers and set $d := \gcd(m, n)$. Prove that the abelian groups $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$ and $\mathbb{Z}/d\mathbb{Z}$ are isomorphic.

5. Let R be a commutative ring and let P be an R-module. Show that the following are equivalent:

(i) P is finitely generated and projective.

(ii) There exist elements $p_1, \ldots, p_n \in P$ and homomorphisms $f_1, \ldots, f_n \in \mathrm{Hom}_R(P, R)$ such that $\sum_{i=1}^{n} f_i(x)p_i = x$ for all $x \in P$.

6. Let V be a 4-dimensional real vector space and let T be an endomorphism of V that is diagonalizable with eigenvalues 1, 2, 3, and 4. Show that also $T \wedge T \colon V \wedge V \to V \wedge V$, $v \wedge w \mapsto T(v) \wedge T(w)$, is diagonalizable and determine the eigenvalues of $T \wedge T$.

7. Find an algebraic number α such that $\mathrm{Gal}(\mathbb{Q}(\alpha)/\mathbb{Q}) \cong \mathbb{Z}/11\mathbb{Z}$.

8. Let p and q be primes. Let $f(x)$ be the product of all monic irreducible polynomials in $\mathbb{F}_p[x]$ of degree q^2. Compute the polynomial $f(x)$ explicitly.

9. Compute the Galois group of the polynomial $f(x) = x^4 - 14x^2 + 9 \in \mathbb{Q}[x]$.