Algebra Qualifying Examination
Spring 2000

1. Let G be a group of order 351. Prove that G has a normal Sylow subgroup.

2. Let G be a group and X a set on which G has a left action and acts transitively.
a) Assume that G is finite and acts transitively on X. Prove that

$$card(X) = |G : G_x|$$

where G_x is the stablizer in G of $x \in X$.
b) Assume that G acts transitively on X and that H is a subgroup of G. Prove that H acts transitively on X if and only if

$$HG_x = G.$$

3. Let R be a unique factorization domain with field of fractions F. Let $f(X)$ be a monic polynomial in $R[X]$ and let $\alpha \in F$ satisfy $f(\alpha) = 0$. Prove that $\alpha \in R$.

4. Let $F = \{0, 1\}$ be a field with two elements.
(a) Show that $f(X) := X^4 + X + 1 \in F[X]$ is irreducible.
(b) How many elements does the field $E := F[X]/(f(X))$ have?
(c) Find an element $\alpha \in E \setminus \{0\}$ of multiplicative order 15.
(d) Give a list of all proper subfields $K \subset E$ together with their elements.

5. (a) Give a complete list of non-isomorphic abelian groups of order 100 which have an element of order 50.
(b) How many subgroups of order 50 does each of the groups in (a) have?
(c) How many elements of order 50 does each of the groups in (a) have.

6. Let K be a field, let $f(X) \in K[X]$ have degree n, and let L be a splitting field of $f(X)$ over K. Show that $[L : K]$ divides $n!$.
(Hint: Use induction on n and distinguish the two cases that $f(X)$ is irreducible or not. Note that $k!(n-k)!$ divides $n!$ for all $k = 1, \dots, n-1$; why?)

7. Let V be a vector space of dimension n over a commutative field. Let $1 < k \leq n$ and assume that $\omega \in \bigwedge^k(V), \omega \neq 0$. Suppose that $Ann(\omega) := \{v \in V | v \wedge \omega = 0\}$ has dimension k. Let $v_1, v_2, \ldots, v_k$ be a basis for $Ann(\omega)$. Prove for some scalar a that $\omega = a(v_1 \wedge v_2 \wedge \cdots \wedge v_k)$.

8. Let A be a free Abelian group of rank $2n$. Let $\langle u, v \rangle = -\langle v, u \rangle$ be a skew symmetric non-degeneate bilinear form on A with values in $\mathbb{Z}$. Show that there exists a basis $a_1, \ldots, a_n, a_{n+1}, \ldots, a_{2n}$ of A such that $\langle a_i, a_{n+i} \rangle = d_i$ where d_i divides d_{i+1} for $i = 1, 2, \ldots, n$.