Algebra Preliminary Examination for Spring 2001
Put each solution on a separate sheet of paper.

Notation: $\mathbb{Z}$ is the set of integers, $\mathbb{Q}$ is the field of rational numbers and $\mathbb{C}$ is the field of complex numbers. For a matrix m, m^t denotes the transpose of m.

1. a) Let G be a group and H a subgroup of G. Define what is meant by the index of H in G.

b) Let G be a group and assume that H, K are subgroups of finite index:

$$[G : H] = h, [G : K] = k; h, k \in \mathbb{N}.$$

Prove that $[H : H \cap K] \leq [G : K] = k$. (Hint: Prove that if h_1, h_2 are distinct coset representatives for $H \cap K$ in H then $h_1 K \neq h_2 K$.)
c) Continuing with the hypothese of b) further assume that $gcd\{h, k\} = 1$. Prove that $G = HK$.

2. Let G be a group of order 56. Prove that some Sylow subgroup of G is normal.

3. a) Let R be a ring. Define what is meant by an ideal I of R.

b) For ideals I, J in the ring R set $I + J = \{a + b : a \in I, b \in J\}$. Prove that $I + J$ is an ideal.

c) For ideals I, J in the ring R let

$$IJ = \{a_1 b_1 + a_2 b_2 + \ldots a_n b_n : a_i \in I, b_i \in J\}.$$

Prove that IJ is an ideal of R.

d) Assume that $I + J = R$. Prove that $IJ = I \cap J$.

4. Let A be a $n \times n$ complex matrix. Prove that A is similar to a diagonal matrix if and only if the minimal polynomial has n distinct roots.

5. Let $a, b \in \mathbb{C}$ be algebraic over $\mathbb{Q}$. Prove that both $a+b$ and ab are algebraic over $\mathbb{Q}$.

6. Let F be a field of characteristic p. Determine the Galois group over F of a splitting field of $x^p - x - a$.

7. Consider the antisymmetric bilinear form

$$\beta\colon \mathbb{Q}^4 \times \mathbb{Q}^4 \to \mathbb{Q}, \quad (x,y) \mapsto x^t \cdot \begin{pmatrix} 0 & -1 & 2 & 0 \\ 1 & 0 & 1 & 3 \\ -2 & -1 & 0 & -1 \\ 0 & -3 & 1 & 0 \end{pmatrix} \cdot y\,.$$

Find a basis (v_1, v_2, v_3, v_4) of $\mathbb{Q}^4$ such that the representing matrix of β with respect to this basis has the form

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

8. Let m and n be positive integers and let d be the greatest common divisor of m and n. Show that $\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z} \cong \mathbb{Z}/d\mathbb{Z}$.

9. Let V be a vector space over a field of characteristic different from 2, let $(e_1, \ldots, e_n)$ be a basis of V and set $\mu := e_1 \wedge \ldots \wedge e_n$. Let $v_1, \ldots, v_n \in V$ be arbitrary elements and let $A = (\alpha_{ij})$ be the matrix defined by $v_j = \sum_{i=1}^{n} \alpha_{ij} e_i$. Show that $v_1 \wedge \ldots \wedge v_n = \det(A) \cdot \mu$.