Algebra Preliminary Exam

Spring 2002

1. (a) State the Sylow Theorems.

   (b) Prove that any group of order 30 is solvable.

2. Prove that if H is a subgroup of a finite group G of index p, where p is the smallest prime dividing the order of G, then H is normal in G.

3. (a) Let R be a PID. Prove that any two elements a, b (not both zero) have a greatest common divisor d and that d can be written as $d = ax + by$ for some $x, y \in R$.

   (b) Prove or disprove the statement in (a) when R is a UFD.

4. (a) State the classification theorem for finitely generated modules over a principal ideal domain. Use it to show that every finite abelian group is the direct product of cyclic groups.

   (b) Make a list of all abelian groups (up to isomorphism) of order $2^3 \cdot 3^4 \cdot 5$ without repetition.

5. Let E be a field extension of F and let $a, b \in E$ be algebraic over F with the same minimal polynomial. Prove that $F(a)$ and $F(b)$ are isomorphic.

6. (a) Determine the splitting field of $x^4 - 2$ over $\mathbb{Q}$.

   (b) Determine the Galois group of $x^4 - 2$ over $\mathbb{Q}$.

7. Let vector space V (over a field) have basis $\{v_1, v_2, v_3, v_4\}$. Let I be the ideal in $\wedge(V)$ generated by $v_1 \wedge v_2 \wedge v_3$ and $v_2 \wedge v_4$. Find the dimension of the quotient ring $\wedge(V)/I$.

8. Let V be a finite dimensional vector space over field K and $f : V \times V \to K$ a nondegenerate symmetric bilinear form on V. Let X be a subspace of V.

   (a) Prove that $\dim X + \dim X^\perp = \dim V$.

   (b) Prove or disprove that $X + X^\perp = V$.