Preliminary Exam, Algebra, Spring 2003

S1. Let G and H be finite groups with coprime orders. Show that every subgroup of the direct product $G \times H$ is of the form $U \times V$ for subgroups $U \leq G$ and $V \leq H$.

S2. Let N be a normal subgroup of the group G. Assume that G and N are solvable. Show that G is solvable. G/N

S3. Compute the greatest common divisor of $a = 1027$ and $b = 480$ in $\mathbb{Z}$ and write it as an integral linear combination of a and b.

S4. Compute the invariant factors of the matrix

$$\begin{pmatrix} 3 & 5 & -1 \\ 2 & -2 & 1 \\ 3 & 4 & 0 \end{pmatrix}.$$

S5. (a) Show that the polynomial $f(X) = X^4 + X + 1 \in \mathbb{Z}/2\mathbb{Z}[X]$ is irreducible.

(b) Set $F := \mathbb{Z}/2\mathbb{Z}[X]/(f(X))$ and compute a generator of the unit group $F^\times$ of F.

S6. Let E/F be a field extension. Show that the set

$$E' := \{x \in E \mid x \text{ is separable over } F\}$$

is a subfield of E that contains F.

S7. Suppose that R is an integral domain and I is a principal ideal of R. Prove that $I \otimes_R I$ is a torsion free R-module.

S8. Let $A \in M_n(\mathbb{C})$ be a Hermitian matrix. Prove that A is positive definite if and only if there exits $P \in GL_n(\mathbb{C})$ such that $A = P^*P$.

S9. Let $A \in M_n(\mathbb{R})$ be a symmetric matrix. Prove that $tI_n + A$ is positive definite if $t \in \mathbb{R}$ is large enough.