1. Let $G$ be a finite group and let $p$ be a prime. An element $x \in G$ is called a \textit{p-element} if its order equals $p^n$ for some $n \in \mathbb{N}_0$, and $x$ is called a \textit{p'-element}, if its order is not divisible by $p$.

Show that for every $x \in G$ there exist unique elements $a, b \in G$ such that $a$ is a p-element, $b$ is p'-element, and $x = ab = ba$.

2. Let $G$ be a group of order 231. Prove that the 11-Sylow subgroup is in the center of $G$.

3. (a) Define the notion of a greatest common divisor in a domain with 1.
(b) Let $D$ be a PID. Prove that for any $a, b \in D$ not both zero, a greatest common divisor $d$ of $a$ and $b$ exists and that there exist $x, y \in D$ such that $d = ax + by$.

4. Let $R$ be a commutative ring, let $I_1, \ldots, I_n$ be ideals of $R$, and let $P$ be a prime ideal of $R$ such that $P$ contains $I_1 \cap \cdots \cap I_n$. Show that $P$ contains one of the ideals $I_1, \ldots, I_n$.

5. Let $R$ be a ring and let $M$ be a left $R$-module. One calls $M$ \textit{noetherian} if every submodule of $M$ is finitely generated.

Let $U$ be a submodule of $M$. Show that $M$ is noetherian if and only if $U$ and $M/U$ are noetherian.

6. Let $F$ be a finite field with 81 elements. Find the number of roots in $F$ of the following polynomials in $F[X]$:
(a) $f(X) = X^{20} - 1$.
(b) $g(X) = X^9 - 1$.
(c) $h(X) = X^{2004} - X^4$.

7. If the field $F$ contains a primitive $n$-th root of unity, prove that the Galois group of $X^n - a$ is abelian.

8. Let $R$ be a ring, $I$ an ideal of $R$, and $M$ a left $R$-module. Let $IM$ be the set of all sums of elements of the form $am$ with $a \in I$ and $m \in M$.

(a) Show that $IM$ is an $R$-submodule of $M$ and that $M/IM$ is not only an $R$-module but also an $R/I$-module.
(b) Show that $M/IM$ and $R/I \otimes_R M$ are isomorphic $R/I$-modules, where $R/I$ is considered as $R/I$-$R$-bimodule in the natural way.

9. Let $m, n$ be positive integers. Prove that the groups $\mathbb{Z}/m\mathbb{Z} \otimes_\mathbb{Z} \mathbb{Z}/n\mathbb{Z}$ and $\mathbb{Z}/d\mathbb{Z}$ are isomorphic, where $d$ is the greatest common divisor of $m, n$. 