Preliminary Exam, Algebra, Spring 2004

1. Let G be a finite group and let p be a prime. An element $x \in G$ is called a *p-element* if its order equals p^n for some $n \in \mathbb{N}_0$, and x is called a *p'-element*, if its order is not divisible by p.

Show that for every $x \in G$ there exist unique elements $a, b \in G$ such that a is a p-element, b is p'-element, and $x = ab = ba$.

2. Let G be a group of order 231. Prove that the 11-Sylow subgroup is in the center of G.

3. (a) Define the notion of a greatest common divisor in a domain with 1.

(b) Let D be a PID. Prove that for any $a, b \in D$ not both zero, a greatest common divisor d of a and b exists and that there exist $x, y \in D$ such that $d = ax + by$.

4. Let R be a commutative ring, let $I_1, \ldots, I_n$ be ideals of R, and let P be a prime ideal of R such that P contains $I_1 \cap \cdots \cap I_n$. Show that P contains one of the Ideals $I_1, \ldots, I_n$.

5. Let R be a ring and let M be a left R-module. One calls M *noetherian* if every submodule of M is finitely generated.

Let U be a submodule of M. Show that M is noetherian if and only if U and M/U are noetherian.

6. Let F be a finite field with 81 elements. Find the number of roots in F of the following polynomials in $F[X]$:

(a) $f(X) = X^{20} - 1$.

(b) $g(X) = X^9 - 1$.

(c) $h(X) = X^{2004} - X^4$.

7. If the field F contains a primitive n-th root of unity, prove that the Galois group of $X^n - a$ is abelian.

8. Let R be a ring, I an ideal of R, and M a left R-module. Let IM be the set of all sums of elements of the form am with $a \in I$ and $m \in M$.

(a) Show that IM is an R-submodule of M and that M/IM is not only an R-module but also an R/I-module.

(b) Show that M/IM and $R/I \otimes_R M$ are isomorphic R/I-modules, where R/I is considered as R/I-R-bimodule in the natural way.

9. Let m, n be positive integers. Prove that the groups $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$ and $\mathbb{Z}/d\mathbb{Z}$ are isomorphic, where d is the greatest common divisor of m, n.