Geometry–Topology Prelim, Spring 2006

1 (5 points). Consider the vector fields

$$X = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} \quad \text{and} \quad X = (y + xy^2)\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$$

on $\mathbb{R}^2$. Do the flows of X and Y commute? Justify your answer.

2 (10 points). Let X be a (complete) vector field on a manifold M and let α be a one-form on M such that $i_X d\alpha = 0$. Denote by φ^t the flow of X. Prove that

$$\int_\gamma \alpha = \int_{\varphi^t(\gamma)} \alpha$$

for any closed curve γ.

3 (15 points). Let $\pi\colon M \to N$ be a submersion with connected fibers and let α be a closed k-form on M such that $i_X\alpha = 0$ for every vector X tangent to a fiber. Prove that there exists a closed k-form β on N such that $\pi^*\beta = \alpha$. Is β necessarily exact, if α is exact?

4 (15 points). Let $M \subset \mathbb{R}^3$ be a closed hypersurface. Show that the curvature of M is positive at some points of M.

5 (15 points). Let $\gamma\colon S^1 \to \mathbb{R}^2$ be an immersion of the circle into the plane. Recall that the rotation number $r(\gamma)$ of γ is the degree of the map $\dot\gamma/\|\dot\gamma\|\colon S^1 \to S^1$. Furthermore, the geodesic curvature of k_g of γ is defined as follows. Assume that γ is parametrized by arc-length s so that $\gamma\colon [0,\, l] \to \mathbb{R}^2$, where l is the length of γ. Then $k_g(s)\nu(s) = \ddot\gamma(s)$, where $\nu(s)$ is the "inner normal" to γ at s (i.e., the frame $\{\dot\gamma(s), \nu(s)\}$ is positive). Prove that

$$\int_0^l k_g(s)\, ds = 2\pi r(\gamma).$$

6 (10 points). Construct an embedding of $S^n \times S^k$ into $\mathbb{R}^{n+k+1}$.

7 (15 points). Let M and N be closed even-dimensional manifolds. Prove that $\chi(M\#N) = \chi(M) + \chi(N) - 2$, where $M\#N$ is the connected sum of M and N. Is the same true when the manifolds are not assumed to be even-dimensional?

8 (15 points). Let M be a smooth manifold such that $\pi_1(M)$ is finite. Denote by $\pi\colon \tilde M \to M$ the universal covering of M. Prove that $\pi^*\colon H^*_{dR}(M) \to H^*_{dR}(\tilde M)$ is a monomorphism. Is it true that π^* is necessarily an isomorphism?