1. (5 points). Consider the vector fields
\[ X = y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y} \quad \text{and} \quad X = (y + xy^2) \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} \]
on \mathbb{R}^2. Do the flows of \( X \) and \( Y \) commute? Justify your answer.

2. (10 points). Let \( X \) be a (complete) vector field on a manifold \( M \) and let \( \alpha \) be a one-form on \( M \) such that \( i_X d\alpha = 0 \). Denote by \( \varphi^t \) the flow of \( X \). Prove that
\[ \int_\gamma \alpha = \int_{\varphi^t(\gamma)} \alpha \]
for any closed curve \( \gamma \).

3. (15 points). Let \( \pi: M \to N \) be a submersion with connected fibers and let \( \alpha \) be a closed \( k \)-form on \( M \) such that \( i_X \alpha = 0 \) for every vector \( X \) tangent to a fiber. Prove that there exists a closed \( k \)-form \( \beta \) on \( N \) such that \( \pi^* \beta = \alpha \). Is \( \beta \) necessarily exact, if \( \alpha \) is exact?

4. (15 points). Let \( M \subset \mathbb{R}^3 \) be a closed hypersurface. Show that the curvature of \( M \) is positive at some points of \( M \).

5. (15 points). Let \( \gamma: S^1 \to \mathbb{R}^2 \) be an immersion of the circle into the plane. Recall that the rotation number \( r(\gamma) \) of \( \gamma \) is the degree of the map \( \hat{\gamma}/\|\hat{\gamma}\|: S^1 \to S^1 \). Furthermore, the geodesic curvature of \( k_\gamma \) of \( \gamma \) is defined as follows. Assume that \( \gamma \) is parametrized by arc-length \( s \) so that \( \gamma: [0, l] \to \mathbb{R}^2 \), where \( l \) is the length of \( \gamma \). Then \( k_\gamma(s) \nu(s) = \hat{\gamma}(s) \), where \( \nu(s) \) is the “inner normal” to \( \gamma \) at \( s \) (i.e., the frame \( \{\hat{\gamma}(s), \nu(s)\} \) is positive). Prove that
\[ \int_0^l k_\gamma(s) \, ds = 2\pi r(\gamma). \]

6. (10 points). Construct an embedding of \( S^n \times S^k \) into \( \mathbb{R}^{n+k+1} \).

7. (15 points). Let \( M \) and \( N \) be closed even-dimensional manifolds. Prove that \( \chi(M \# N) = \chi(M) + \chi(N) - 2 \), where \( M \# N \) is the connected sum of \( M \) and \( N \). Is the same true when the manifolds are not assumed to be even-dimensional?

8. (15 points). Let \( M \) be a smooth manifold such that \( \pi_1(M) \) is finite. Denote by \( \pi: \tilde{M} \to M \) the universal covering of \( M \). Prove that \( \pi^*: H^*_{dR}(M) \to H^*_{dR}(\tilde{M}) \) is a monomorphism. Is it true that \( \pi^* \) is necessarily an isomorphism?