Algebra Preliminary Exam **Spring 2007**

Instructions:

1. Please take a minute to write your name on *every* page of the exam. This is important, because the exams are going to be separated and reassembled during the grading process!

2. Please ask if you need more scratch paper.

3. Each question is worth five points. Among these points:

   - Two points will measure your *understanding of basic concepts.*
   - One point will measure your computational competence (you will only lose a point for a careless computational error).
   - Two points will assess the clarity and correctness of your proofs and explanations.

4. You may not use any outside sources or notes. Feel free to take bathroom/water breaks when you want, but please leave your papers in the room if you leave.

Question:	1	2	3	4	5	6	7	8	9	Total
Points:	5	5	5	5	5	5	5	5	5	45
Score:										

1. (5 points) Let p be a prime and let G be a p-group. Let $Z(G)$ denote the center of G. Show the following:
   (a) If G has order p^2 then G is abelian.
   (b) If N is a normal subgroup of G and $N \neq \{1\}$, then $Z(G) \cap N \neq \{1\}$.

2. (5 points) Let $n > 1$ be a natural number. For $i \in \{1, \ldots, n\}$ define H_i as the subset of $G := \mathrm{Sym}(n)$ consisting of permutations σ with $\sigma(i) = i$.

   (a) Prove that $H_1, \ldots, H_n$ are pairwise conjugate subgroups of G.

   (b) Prove that $N_G(H_i) = H_i$ for all $i = 1, \ldots, n$.

   (c) Prove that H_i is a maximal subgroup of G for all $i = 1, \ldots, n$.

3. (5 points) Let R be the ring $\mathbb{Z}/2007\mathbb{Z}$.
   (a) Determine the number of units of R.
   (b) Is the unit group of R cyclic?

4. (5 points) Let V be a $\mathbb{F}_q$ vector space of finite dimension $n \geq 2$, where $\mathbb{F}_q$ is a finite field containing q elements.

   (a) Let S be the set of ordered pairs (v_1, v_2), where $v_1, v_2 \in V$, and $Span(v_1, v_2)$ is two-dimensional. Prove that:

   $$\#S = (q^n - 1)(q^n - q).$$

   (b) Prove that the number of two-dimensional subspaces of V is equal to:

   $$\frac{(q^n - 1)(q^n - q)}{(q^2 - 1)(q^2 - q)}.$$

5. (5 points) Let A be an $n \times n$ matrix with entries in the field of complex numbers. Recall that A is called *nilpotent* if there is a positive integer k such that $A^k = 0$. Prove that the following are equivalent:

   - A is nilpotent
   - $\text{Trace}(A^m) = 0$ for all integers $m \geq 0$.

6. (5 points) Consider the diagram of finite-dimensional K-linear spaces (K a field) and K-linear maps between them:

$$0 \xrightarrow{f_0} V_1 \xrightarrow{f_1} ..... \xrightarrow{f_{i-1}} V_i \xrightarrow{f_i} ..... \xrightarrow{f_{n-1}} V_n \xrightarrow{f_n} 0.$$

Assume that $\mathrm{im} f_i = \ker f_{i+1}$ for all indices i. Prove that

$$\sum_{i=1}^{n} (-1)^i \dim V_i = 0.$$

(Hint: use induction on n.)

7. (5 points) Suppose that R is a commutative ring, and M is a R-module.

   Suppose that I, J are distinct maximal ideals of R. Suppose that $m, n \in M$. Suppose that, for all $i \in I$, all $j \in J$, we have:

$$i \cdot m = 0 \text{ and } j \cdot n = 0.$$

   Prove that $m \otimes n = 0$ in $M \otimes_R M$.

8. (5 points) Let K be the field $\mathbb{Q}(\zeta)$, where ζ denotes a primitive fifteenth root of unity in $\mathbb{C}$.

   (a) Draw a diagram, illustrating all of the subfields of K, containments among them, and degrees of extensions. (You don't have to explicitly describe the fields...yet).

   (b) Describe explicitly the subfield(s) of K, which are quadratic as extension(s) of $\mathbb{Q}$. In other words, describe them as $\mathbb{Q}(\sqrt{a})$ for some integer(s) a.

9. (5 points) Suppose that F is a field, which contains a primitive cube root of unity (in particular, F is does not have characteristic 3). Suppose that K/F is a finite extension of the form $K = F(a, b)$ where $a, b \notin F$, $a^3 \in F$ and $b^3 \in F$, and $b \notin F(a)$.

   (a) Prove that K is a Galois extension of F, and $Gal(K/F)$ is isomorphic to $(\mathbb{Z}/3\mathbb{Z}) \times (\mathbb{Z}/3\mathbb{Z})$.

   (b) How many intermediate fields are there, between F and K? Justify your answer.