# Preliminary Exam, Algebra, June 5, 2009

**1.** Show that the group $G = \langle s, t \mid s^2 = t^2 = (st)^3 = 1 \rangle$ is isomorphic to the symmetric group $\mathrm{Sym}(3)$.

**2.** (a) Show that every finite integral domain is a field.
(b) Show that the center of a division ring is a field.

**3.** Let $R = \{a + bi \mid a, b \in \mathbb{Z}\}$ be the ring of Gaussian integers. Show that $I := (2 + i)R$ is a maximal ideal of $R$.

**4.** Let $K$ be a field and $V$ a finite-dimensional $K$-linear space.
(a) Explain the construction (without detailed proofs) of the *exterior algebra* $\Lambda(V)$ regarded as $\mathbb{Z}$-graded $K$-algebra.
(b) Prove that $\Lambda(V)$ is a *local algebra* in the following sense: the subspace $J \subseteq \Lambda(V)$ spanned by elements of *positive degree* is a nilpotent ideal of codimension 1, and every element in $\Lambda(V) \setminus J$ is a unit.

**5.** (a) Let $A$ be an $n \times n$ matrix with entries in the complex numbers $\mathbb{C}$. Show that $A$ is *similar* to an *upper triangular* matrix.
(b) Show that (a) is *false* if $\mathbb{C}$ is replaced by $\mathbb{R}$.

**6.** Let $K$ be a field, $V$ a $K$-linear space, and $f\colon V \to V$ a $K$-linear transformation. Suppose that $v_i \in V$ is a family of *non-zero* vectors ($I$ an index set) such that $f(v_i) = \lambda_i v_i$ with scalars $\lambda_i \in K$ such that $\lambda_i \neq \lambda_j$ whenever $i \neq j$. Prove that $\{v_i\}_{i \in I}$ is a *linearly independent* set.

**7.** Let $U = \{(a, b, c, d) \in \mathbb{Z}^4 : a \equiv b \pmod 3, 2(c + d) = 5a\}$.
(a) Show that $U$ is a free abelian group, and compute a basis for $U$.
(b) Compute the structure of the quotient $\mathbb{Z}^4/U$.

**8.** Suppose that $R$ is a (commutative) integral domain. Suppose that $I$ and $J$ are ideals of $R$ such that $IJ$ is a non-zero principal ideal. Prove that $I$ and $J$ are finitely generated ideals.

**9.** (a) Compute the Galois group of $x^5 - 10x + 5$ over $\mathbb{F}_3$.
(b) Compute the Galois group of $x^5 - 10x + 5$ over $\mathbb{Q}$.