Algebra Preliminary Exam

Spring 2010

1. Let G and H be isomorphic groups. Show that also $G/Z(G)$ and $H/Z(H)$ are isomorphic.

2. Show that every group of order 36 is solvable.

3. Let R be a ring. An element $r \in R$ is called *nilpotent* if there exists a positive integer n such that $r^n = 0$.

(a) Show that if r is nilpotent then $1 - r$ is a unit of R.

(b) Show that if R is commutative then the set N of nilpotent elements of R is an ideal of R.

(c) Assume that R is commutative and let N be the ideal of nilpotent elements of R. Show that R/N has precisely one nilpotent element.

4. Let A be a real symmetric matrix. Prove that e^A is positive definite.

5. Let $A, B \in M_n(F)$ such that $AB = BA$ and A, B are diagonalizable. Prove that there is C such that CAC^{-1} and CBC^{-1} are diagonal. Also prove that the result is false if $AB \neq BA$.

6. Let W be a subspace of V invariant under $T \in A(V)$. Denote the restriction of T to W by T_1 and the induced linear transformation of T on V/W by T_2. Let $m(x), m_1(x)$ and $m_2(x)$ be the minimal polynomials of T, T_1, T_2, respectively.

(a) Prove that $m_1(x)|m(x)$.

(b) prove that $m_2(x)|m(x)$.

(c) prove that $m(x)|m_1(x)m_2(x)$.

(d) If $m_1(x)$ and $m_2(x)$ are coprime, prove that $m(x) = m_1(x)m_2(x)$,

(e) Give an example of T for which $m(x) \neq m_1(x)m_2(x)$.

7. (a) State the Fundamental Theorem of Galois Theory.

(b) Call an extension of fields E/F *biquadratic* if there are $D_1, D_2 \in F$ such that $E = F(\sqrt{D_1}, \sqrt{D_2})$ and $|E : F| = 4$. Using the Fundamental Theorem (or otherwise) show that a *finite* field F does *not* have any biquadratic extensions.

8. Let R be a commutative ring.

(a) Define a finitely generated (f.g.) *free* R-module.

(b) Prove that a f.g. free R-module is *projective*. Ie., if $\varphi : M \to F$ is a surjective morphism of R-modules such that F is f.g. and free, then there is an R-subodule $N \subseteq M$such that $M \cong N \oplus F$.

9. Let A be a finite abelian p-group (p a prime), with $A_p = \{a \in A \mid pa = 0\}$, $pA = \{pa \mid a \in A\}$.

(a) Prove that A_p and A/pA are *isomorphic* groups.

(b) Suppose that $|A_p| = p^n$. Prove that n is the number of elementary divisors of A.