Algebra Preliminary Exam Name ______________________
Spring, 2011

1. Let G be a finite group of order p^2q^2 where p, q are primes and $p > q$.
   (a) If G has odd order, prove that a Sylow p-subgroup of G is *normal*.
   (b) Prove that, regardless of whether G has even or odd order, *some* Sylow subgroup of G is normal.

2. Let F be a field.
   (a) Explain why $F[x]$ is a *principal ideal domain*.
   (b) Prove that $F[x, y]$ is *not* a principal ideal domain.

3. Let R be a commutative ring and $D \subseteq R \setminus \{0\}$ a multiplicative subset.
   (a) Describe the *localization* $D^{-1}R$.
   (b) If R is a PID (*principal ideal domain*), prove that $D^{-1}R$ is also a PID.

4. Let $\mathbb{F}_7$ be the field with 7 elements.
   (a) What is the cardinality of $SL_2(\mathbb{F}_7)$?
   (b) Assume that $g \in SL_2(\mathbb{F}_7)$ and $g^7 = 1$. What is the minimal polynomial of g?

5. Let $g \in GL_n(\mathbb{R})$ be an invertible matrix satisfying $gg^t = 1_n$, where g^t denotes the transpose of g, and 1_n is the identity matrix. Prove that, within the larger group $GL_n(\mathbb{C})$, g is conjugate to a diagonal matrix. Prove moreover that if $\lambda \in \mathbb{C}$ is an eigenvalue of g, then $|\lambda| = 1$.

6. Let $\phi : V \to V$ be a linear operator on a finite-dimensional vector space V. Let $k = dim(Ker(\phi))$ and let $d = dim(V)$. Let $\phi^2 : V \otimes V \to V \otimes V$ be the unique linear operator which satisfies $\phi^2(v_1 \otimes v_2) = \phi(v_1) \otimes \phi(v_2)$ for all $v_1, v_2 \in V$. Prove that $dim(Ker(\phi^2)) = 2dk - k^2$.

7. Let A be a nonzero finite abelian group.
   (a) Prove that A is not a projective $\mathbb{Z}$-module.
   (b) Prove that A is not an injective $\mathbb{Z}$-module.

8. Let L/K be a Galois extension of fields, with $\text{Gal}(L/K) = \{\sigma_1, \sigma_2, \ldots, \sigma_n\}$. Let $\alpha \in L$. Show that $L = K(\alpha)$ if and only if $\sigma_1(\alpha), \sigma_2(\alpha), \ldots, \sigma_n(\alpha)$ are distinct.

9. Let p be a prime number and let ζ denote a primitive p^{th} root of unity in $\mathbb{C}$. Show that $\mathbb{Q}(\zeta + \zeta^{-1})$ is Galois over $\mathbb{Q}$ with cyclic Galois group. What is the size of this Galois group?