Algebra Prelim, Spring 2012

1. Let $G_1 := \mathrm{Alt}(4)$ be the alternating group on 4 elements, let $G_2 := D_{12}$ be the dihedral group of order 12, and let $G_3 := C_3 \rtimes C_4$ denote the semidirect product of a cyclic group $C_3 = \langle x \rangle$ of order 3 and a cyclic group $C_4 = \langle y \rangle$ of order 4, with the action of C_4 on C_3 determined by ${}^{y}x := x^{-1}$. Show that these three non-abelian groups of order 12 are pairwise non-isomorphic.

2. Let G be a group such that $G/Z(G)$ is cyclic. Show that the derived subgroup (commutator subgroup) of G is trivial.

3. Let R be a commutative ring and let $S \subseteq T \subseteq R$ be two multiplicatively closed subsets with $1 \in S$. We consider the localizations $S^{-1}R$ and $T^{-1}R$ of R.

(a) Let $r \in R$ and $s \in S$. Show that $r/s = 0$ in $S^{-1}R$ if and only if there exists $s' \in S$ such that $s'r = 0$ in R.

(b) Show that there exists a ring homomorphism $\alpha\colon S^{-1}R \to T^{-1}R$ with the property that $\alpha(r/s) = r/s$.

(c) Show that if each element of T is a divisor of an element of S then the homomorphism α in (a) is an isomorphism.

4. Let V be a finite dimensional vector space over a field $\mathbb{F}$ and let T be an $\mathbb{F}$-linear operator of V.

For $v \in V$ let $\langle T, v \rangle$ denote the $\mathbb{F}$-span of the set $\{T^i(v) \mid i \geq 0\}$ and let $\mu_{T,v}(x) = x^d + \alpha_{d-1}x^{d-1} + \cdots + \alpha_0 \in \mathbb{F}[x]$ denote the unique monic polynomial of minimal degree such that $T^d(v) + \alpha_{d-1}T^{d-1}(v) + \cdots + \alpha_0 v = 0$.

Assume that T is not a cyclic operator, i.e., that for all $v \in V$ one has $\langle T, v \rangle \neq V$.

(a) Prove that there is an irreducible polynomial $p(x) \in \mathbb{F}[x]$ and vectors $v, w \in V$ which satisfy $\mu_{T,v}(x) = p(x) = \mu_{T,w}(x)$ and $\langle T, v \rangle \cap \langle T, w \rangle = \{0\}$.

(b) Assume that $\mathbb{F}$ is infinite. Prove that there are infinitely many T-invariant subspaces of V.

5. Let V be a six-dimensional vector space over a field $\mathbb{F}$ with basis $(v_1, \ldots, v_6)$. Let the map $\beta\colon \bigwedge^3(V) \times \bigwedge^3(V) \to \mathbb{F}$ be defined by the equation $u \wedge w = \beta(u, w)(v_1 \wedge v_2 \wedge \ldots \wedge v_6)$ for any $u, w \in \bigwedge^3(V)$.

Prove that β is a non-degenerate alternate bilinear form on $\bigwedge^3(V)$.

6. Let A be a symmetric $n \times n$-matrix with real entries. Show that the following are equivalent:

(i) For every non-zero $x \in \mathbb{R}^n$ one has $x^t A x > 0$;

(ii) All the eigenvalues of A are positive;

(iii) There exists an invertible matrix Q such that $A = QQ^t$.

7. Let R be a commutative ring, and let $a \in R$. Prove that the following are equivalent:

(i) $a \in a^2 R$

(ii) aR is an R-module direct summand of R.

(iii) R/aR is a flat R-module.

Note: the simplest method is to prove (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i), and you will get partial credit for any of these implications. Hint for (i) $\Rightarrow$ (ii): show that if $a = a^2 y$, then $R = aR \oplus (1 - ay)R$. Hint for (ii) $\Rightarrow$ (iii): if $aR \oplus B = R$, then $R/aR \cong B$ is projective (why?). Hint for (iii) $\Rightarrow$ (i): tensor the injection $aR \to R$ with R/aR over R.

8. Prove that an $n \times n$ matrix A over $\mathbb{C}$ satisfying $A^3 = A$ can be diagonalized. Give an example to show that this is not true if $\mathbb{C}$ is replaced by an arbitrary field.

9. Show that the polynomials $f(x) = x^6 - 3$ and $g(x) = x^6 + 3$ are irreducible over $\mathbb{Q}$, and find the Galois groups of their splitting fields.