Algebra Prelim, Spring 2013

1. Let $n \geq 3$ be a natural number and let $D_{2n} = \langle \sigma, \tau \mid \sigma^n = \tau^2 = 1, \tau\sigma\tau = \sigma^{-1} \rangle$ denote the dihedral group of order $2n$. Show that $Z(D_{2n}) = \{1\}$ if n is odd and that $Z(D_{2n}) = \{1, \sigma^{n/2}\}$ if n is even.

2. Show that the ideal $I = (3, 2X)$ of the ring $R = \mathbb{Z}[X]$ is not principal and show that R/I is isomorphic to the ring $\mathbb{Z}/3\mathbb{Z}$.

3. Let L/K be a Galois extension of degree 100. Show that there exists a strictly ascending chain of intermediate fields,

$$K = K_0 \subset K_1 \subset K_2 \subset K_3 \subset K_4 = L\,,$$

such that K_i/K_{i-1} is a Galois extension, for $i = 1, 2, 3, 4$.

4. Let V be a finite dimensional vector space (over a field $\mathbb{F}$) with a basis $\mathcal{B} = (v_1, \ldots, v_n)$. Assume that T is an indecomposable operator on V. Prove that there exists an $i \in \{1, \ldots, n\}$ such that $V = \langle T, v_i \rangle$. (Recall that, for an operator T on a finite dimensional vector space V and a vector $v \in V$, $\langle T, v \rangle$ is the subspace of V spanned by $\{v\} \cup \{T^j(v) : j \in \mathbb{N}\}$.)

5. Let $(V, \langle\ ,\ \rangle_V)$ and $(W, \langle\ ,\ \rangle_W)$ be finite dimensional inner product spaces and let T be a linear transformation from V to W. Denote by T^* the adjoint map from W to V. Prove that $\mathrm{Ker}(T^*) = \mathrm{Range}(T)^{\perp}$.

6. Let V be a real 6-dimensional vector space and assume that $T\colon V \to V$ is an operator with eigenvalues 2, 3, 5, 7, 11, and 13. Prove for $1 \leq k \leq 5$ that $\wedge^k(T)\colon \wedge^k(V) \to \wedge^k(V)$ is a diagonalizable cyclic operator.

7. Let R and S be rings. Let M be an (S, R)-bimodule that is flat as a right R-module. Let I be an injective left S-module. Prove that the left R-module $N = \mathrm{Hom}_S(M, I)$ is injective, where the left R-module structure on N is defined by $(rf)(m) := f(mr)$.

8. Let V denote the complex vector space (of dimension $n+1$) of polynomials in $\mathbb{C}[x]$ with degree less than or equal to n. Let $T\colon V \to V$ denote the linear map given by the derivative: $T(f(x)) = f'(x)$.

(a) Find the rational canonical form of T.

(b) Find the Jordan canonical form of T.

9. Let p be a prime number, and denote by $K = \mathbb{F}_p(t)$ the field of rational functions over the finite field of p elements. Let n be a positive integer such that $p \nmid n$, and let $f(x) = x^{p^n} - x - t \in K[x]$. For this problem you may assume without proof that $f(x)$ is irreducible.

(a) Prove that $f(x)$ is separable.

(b) Find the Galois group of the splitting field of $f(x)$ over K.

(Hint: Show that if α is a root of $f(x)$ in its splitting field, then the other roots are $\alpha + \beta$ for $\beta \in \mathbb{F}_{p^n}$.)