Questions in Algebra for Preliminary Exam (Spring 2015)

1. Prove that a group of order 24 without any element of order 6 is isomorphic to S_4.

2. Let R be a commutative ring with 1. Let I, J be comaximal ideals of R, that is, $I + J = R$. Prove that for any positive integer n, I^n and J^n are comaximal and $R/(IJ)^n$ is isomorphic to $R/I^n \times R/J^n$.

3. Let R be a commutative ring without nonzero nilpotent elements. Prove that if $f(x) = \sum_{i=0}^{m} a_i x^i \in R[x]$ is a zero divisor, then there exists $0 \neq b \in R$ such that $bf(x) = 0$.

4. Suppose that a real linear operator f on a 2-dimensional vector space V has trace 2 and determinant 4. Compute the trace and the determinant of the operator $\mathrm{Sym}^2(f)$.

5. Let F be an arbitrary field and $n \geq 1$ an integer. Consider the set of matrices

$$X = \{A \in M_{n\times n}(F) : A^2 = A\},$$

on which the group $G = GL_n(F)$ acts by conjugation:

$$g.A = gAg^{-1} \quad \text{for all } g \in G \text{ and } A \in X.$$

(a) Show that any element A in X is diagonalisable.

(b) Find (with proof) a set of representatives of the orbit space $G\backslash X$.

6. Let $n \geq 1$ be an integer, A a Hermitian $n \times n$ matrix, and B a skew Hermitian $n \times n$ matrix. Show that the real part of $\mathrm{Tr}(AB)$ is zero.

7. Let K/F be an algebraic extension of fields. Let R be a ring such that $F \subset R \subset K$. Prove that R is a field.

8. Let p be a prime number. Prove that the Galois group of $x^p - 2$ over $\mathbb{Q}$ is isomorphic to the group of 2×2 matrices

$$\left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \in \mathbb{F}_p^\times, b \in \mathbb{F}_p \right\}.$$

9. Prove that a module over a ring is projective if and only if it is a direct summand of a free module.