Algebra Preliminary Exam, May 27, 2016

1. Let G be a finite group, let p be a prime, and let P be a Sylow p-subgroup of G. Moreover, let N be a normal subgroup of G. Show that $N \cap P$ is a Sylow p-subgroup of N and that PN/N is a Sylow p-subgroup of G/N. Is the first statement still true if N is replaced with an arbitrary subgroup H of G?

2. Show that the alternating group $\mathrm{Alt}(5)$ has no subgroup of order 15.

3. Consider the subring $R = \{a + b\sqrt{5}i \mid a, b \in \mathbb{Z}\}$ of $\mathbb{C}$. Show that the number 6 is not a product of prime elements in R.
Hint: Use the multiplicative map $N \colon R \to \mathbb{Z}, r = a + b\sqrt{5}i \mapsto r\overline{r} = a^2 + 5b^2$.

4. Let p be a prime number, $\mathbb{F}_p$ the finite field with p elements, and $K = \mathbb{F}_{p^2}$ the degree 2 extension of $\mathbb{F}_p$. Regard the Frobenius endomorphism on K

$$f(x) = x^p$$

as a linear operator on the 2-dimensional $\mathbb{F}_p$-vectorspace K. Compute the characteristic polynomial of f.

5. Prove, or disprove with a counterexample, the following statement: If A and B are $n \times n$ complex matrices, and if A is skew-Hermitian and B is Hermitian, then $\mathrm{Tr}(AB)$ is purely imaginary.

6. Supose that A is a real 2×2 matrix, and that A and $\mathrm{Sym}^2(A)$ have trace 3 and 7, respectively. Prove that A is diagonalisable, and compute the trace of $\mathrm{Sym}^3(A)$.

7. Let n be an integer such that $n \equiv 1 \pmod 6$. Calculate the Galois group of $x^4 + nx + n$ over $\mathbb{Q}$.

8. Let L/K be a Galois extension of fields with $\mathrm{Gal}(L/K) \cong \mathbb{Z}/4\mathbb{Z}$. Show that there exist $a, b \in K$ satisfying the following properties: (1) L is the splitting field of the polynomial $(x^2 - a)^2 - b$, (2) $\sqrt{b} \notin K$, (3) $\sqrt{a^2 - b} \in K(\sqrt{b})$.

9. Let R be a ring and P an R-module.

(a) Define what it means for P to be a projective R-module.

(b) Prove that a free R-module is a projective R-module.

(c) Give an example of a ring R and an R-module P that is projective but not free.