Algebra Preliminary Exam, Spring 2018

1. Assume that G is a simple group of order 168. How many elements of order 7 are there in G?

2. Assume that G is a group (not necessarily finite) which has a proper subgroup of finite index. Show that G has also a proper *normal* subgroup of finite index.

3. Construct explicitly a field with 8 elements and find a generator of its unit group.

4. Prove that if M is a finitely generated non-zero abelian group, then $M \otimes_{\mathbb{Z}} M$ is non-zero. Give a counterexample if M is not finitely generated.

5. Let k be an algebraically closed field and let V be a finite dimensional k-vector space. Let S and T be commuting k-linear endomorphisms of V. Prove that S and T have a common non-zero eigenvector.

6. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$. Write $\mathbb{Z}^3/A\mathbb{Z}^3$ as a product of cyclic groups.

7. Let k be a field and let t be an indeterminate. The group $\mathrm{PGL}(2,k)$ acts on $k(t)$ by fractional linear transformations of the variable t, $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot t = \frac{at+b}{ct+d}$.

(a) Prove that every element of $\mathrm{Aut}(k(t)/k)$ acts on t by a fractional linear transformation.

(b) If k is a finite field, prove that there does not exist an infinite sequence of fields $k(t) \supsetneq L_1 \supsetneq L_2 \supsetneq \dots$ containing k such that $k(t)/L_j$ is Galois for every j.

8. Let ζ_{10} be a primitive 10-th root of unity over the field $\mathbb{Q}$. Let $K = \mathbb{Q}(\sqrt[10]{5}, \zeta_{10})$ be the splitting field of the polynomial $x^{10} - 5 \in \mathbb{Q}[x]$.

(a) Compute $[K : \mathbb{Q}]$.

(b) Find $1 \le j \le 10$ and $1 \le k \le 10$ where $\gcd(k, 10) = 1$ such that the assignment $\sqrt[10]{5} \mapsto \zeta_{10}^j \sqrt[10]{5}$ and $\zeta_{10} \mapsto \zeta_{10}^k$ does **NOT** extend to an automorphism of $K/\mathbb{Q}$.

(c) Find generators and relations for the Galois group $G = \mathrm{Gal}(K/\mathbb{Q})$.

9. Let $f(x) \in \mathbb{Q}[x]$ be an irreducible cubic polynomial with splitting field K. Prove that $G = \mathrm{Gal}(K/F)$ is isomorphic to $\mathbb{Z}/3\mathbb{Z}$ if and only if $D = \mathrm{disc}(f)$ (the discriminant of the polynomial f) is a square in $\mathbb{Q}^\times$.