Algebra Preliminary Exam, Spring 2019

P 1. Let G be a non-trivial finite group and let p be the smallest prime divisor of $|G|$. Prove that any subgroup $H \leq G$ of index p is normal.

P 2. Let G be a non-trivial finite p-group. Prove that the center of G is non-trivial: $Z(G) \neq 1$.

P 3. Let R be a noetherian UFD in which every maximal ideal is principal. Prove that R is a PID.

P 4. Let $M = \mathbb{Z}^2$ and let N be the $\mathbb{Z}$-submodule of M generated by the 2 elements

$$\alpha := \begin{bmatrix} -6 \\ 6 \end{bmatrix} \quad \text{and} \quad \beta := \begin{bmatrix} 8 \\ 12 \end{bmatrix}.$$

(a) Compute the rank and the invariant factors of $Q := M/N$.
(b) Compute the invariant factors of the tensor product $Q \otimes \mathbb{Z}/6\mathbb{Z}$.

P 5. Let V be a 3-dimensional vector space over any field F.
Prove or disprove: For any $\omega \in \bigwedge_F^2 V$, there exist $v_1, v_2 \in V$ such that $\omega = v_1 \wedge v_2$.

P 6. Let $A = \mathbb{R}[x, y, z]/(x^2 + y^2 + z^2 - 1)$ be the ring of real polynomials in 3 variables modulo the principal ideal generated by $x^2 + y^2 + z^2 - 1$. Let $\phi : A^3 \to A$ be the A-module homomorphism given by

$$\begin{bmatrix} f \\ g \\ h \end{bmatrix} \mapsto xf + yg + zh$$

Prove or disprove: $\ker(\phi)$ is a projective A-module.

P 7. Let K/F be a finite separable extension, and let $\varphi \colon F \longrightarrow \overline{F}$ be an embedding into an algebraic closure of F. Determine the number of distinct embeddings $\sigma \colon K \longrightarrow \overline{F}$ extending φ.

P 8. Let K be a splitting field of $f(x) = x^{13} - 2$ over $\mathbb{Q}$. Compute the Galois group $G = \mathrm{Aut}(K/\mathbb{Q})$.
Let $K/\mathbb{Q}$ be any Galois extension whose Galois group has order $2^2 \cdot 3 \cdot 13$. Show that there is an intermediate field E, $\mathbb{Q} \subseteq E \subseteq K$, such that $[E : \mathbb{Q}] = 12$.

P 9. Let q be a prime power, and $f(x) \in \mathbb{F}_q[x]$ an irreducible polynomial of degree d dividing n. Show that $f(x)$ splits completely over $\mathbb{F}_{q^n}$.
Show also that $x^{q^n} - x$ is the product of all monic irreducible polynomials of degree $d \mid n$ in $\mathbb{F}_q[x]$.